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A dynamic model of the firm--estimated and applied

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A DYNAMIC MODEL OF THE FIRM--ESTIMATED AND APPLIED

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I. INTRODUCTION

That the neo-classical literature of micro-economics does not provide a single accepted dynamic model for the firm is well known. Instead, one finds numerous models for nearly every type of market structure that has been conceived. In chapter two of this study a dynamic model of the firm is presented. In this model emphasis is given to input and output inventories, levels of production, and the uses of labor, materials, and machinery. Allowance is made for multi-product, multi-purpose, multi-input firms. Hypothetical representations for the firm's learning about production levels and purchasing materials are offered.

After this model was constructed, data were sought and obtained to see if the hypothetical formulation could withstand the test of application. The null hypothesis of this effort was that the model would stand up with certain modifications. The results of this simple application are provided in chapter five. Along with this application, the methods that have been used to determine the parameters are delineated.

If the model is applicable, there is much more that can be done. Besides estimating the parameters, the model can be implemented as the constraints of numerous optimization problems. Some examples of these have been described in chapter three, and solutions are illustrated in chapter six. In chapter four some of the methods that could be applied in determining the solutions are surveyed. Some of these same methods will be utilized when the parameters are being estimated in chapter five. An example is the use of quadratic programming to find the marginal products for the linear production functions.

This brief description of what will be done does not provide a framework within which the usefulness of this study might be considered. The reader is presumed to be a sophisticated judge of whether the end result is a satisfactory achievement of the target; however, it is the duty of the writer to specify the target. The purposes of the study are to construct an operational dynamic model for the firm and to define objective functions which represent the goals to which the firm is likely to aspire. If such a model and function clarify the analysis and decision making within the firm, they should be considered a useful effort. This may be true even if the optimal revenues, costs, and profits are not the actual achievement of the firm.

Because there are many decisions to be consummated by firm's executives that are not included in this model, the solutions may be inapplicable. However, these solutions could motivate corporate officials to consider how or if these levels might be attained. In this context topics such as capital budgeting are relevant to this study although they are not explicitly discussed here. Also to be considered among the conclusions is the role of public policy with regard to firms to which the model may be applicable.

II. THE THEORETICAL MODEL

A. Introduction

The model to be constructed in this study is intended to be one in micro-economics within the scope of the theory of the firm. Both input and output inventories will be considered in a short run model for the firm. This is in contradistinction to most inventory models found in the operations research literature; too often those studies ignore the behavioral aspects and economic consequences of the included functions.

Before continuing, the distinction between economic theories and models will be described; the difference is discussed by Andreas Papandreou in Economics as a Science (67). Papandreou argues that a theory intends to delineate which activities or variables are interrelated without specifying the actual relationship. For instance, part of a theory might be the existence of a production function, $q = f(x)$, where q is the output and x is a vector of inputs. This implies that observations for each of the elements of x will determine at least one level of output. However, a specification such as $q = 2 + .3\,x_1 + .4\,x_2$ provides an explicit relationship with the values .3 and .4 for the marginal products. An equation of this form would be part of both a theory and a model since $q = 2 + .3\,x_1 + .4\,x_2$ implies $q = f(x)$. Therefore, it can be said that the development of a model implies the existence and development of a theory. The implication is not necessarily reversible.

A model is a specific set of rules which describe the possible effects from employing activities at particular levels. These rules will be represented by a set of equations and inequalities; there may be more relationships than rules or vice versa. A model does not indicate that a particular level of a variable or activity is more desirable than another level. Such an ordering is not possible until a welfare or objective function is introduced.

The role that inventory control rules play in neo-classical studies of the firm is hardly significant. It is not unusual for inventory relationships to be completely ignored. The reason for this may be that most models that include consideration of storage and shortage of materials and outputs have dynamic and stochastic properties. A decision at any level of the firm very much affects the decisions to be consummated at another level of the firm in that and future time periods. The nature of the problem should be more clear after looking at Figure 1.

One stochastic element of this problem may be consumer demand. Holding inventories of outputs and materials as well as the production decisions are dynamic issues which are directly influenced by anticipated consumer demand and delivery practices of the materials' suppliers. Because of these dynamic considerations, nearly every activity of the firm ought to be studied in a dynamic context.

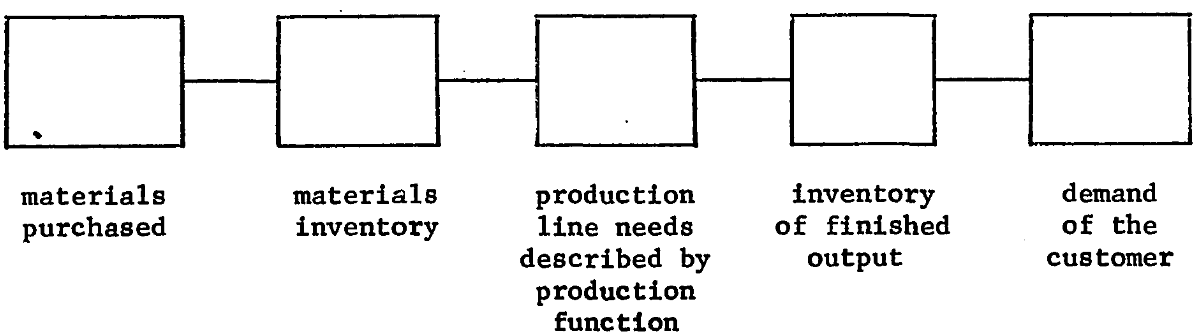

Figure 1. An illustration of a simple production chain

A dynamic model with stochastic elements is to be constructed to represent the situation pictured in Figure 1. This will be a behavioral model in contrast to a computer model that simulates a firm's behavior as offered by Cyert and March (16). A firm shall be defined as any organization that purchases factors of production from a market system, transforms the factors, and returns them in the form of finished goods to any market system. All of these activities shall be occuring simultaneously with the allocation of the various resources at the disposal of the firm. The firm shall be assumed to have a multitude of goals to consider while operating in this context.

The studies that provide the essential background to this one are: Price, Output, and Inventory Policy (59) by Mills; Planning, Production, Inventories, and Work Force (46) by Holt, Modigliani, Muth, and Simon; and Studies in the Mathematical Theory of Inventory and Production (2) by Arrow, Karlin, and Scarf. To complete a thorough review of the literature on inventory and production models would require a survey of several volumes. Even to mention all of the major theoretical and empirical studies of the firm would be a copious task. Instead the model to be considered in this study will be constructed, and the literature that directly pertains to this model will be mentioned.

Three elements of the problem to be studied will be emphasized. The significance of this inquiry will rest on their consideration. These are: (1) the inclusion of a production function; (2) the recognition of dynamic behavioral and learning processes within the firm; and (3) the recognition that a firm has a multitude of goals and purposes. Only the first of these is commonly found in neo-classical analyses of the firm. Perhaps the only economic studies that have emphasized most of these factors are really applications of control theory. Two of these are Mills' Price, Output, and Inventory Policy (59) and Simon's essay On the Application of Servomechanism Theory to the Study of Production Control (83).

For the convenience of the reader the main variables are defined below. Matrix notation will be used wherever the connotation is clear. There are assumed to be n outputs; therefore, $\bar{q}$, q, q^*, q^d, q^s, q^u, and p represent column vectors having n elements. The i th element of each vector is the relevant variable for the i th product. All are presumed to be in the t th time period unless otherwise indicated.

$\bar{q}$ represents the output quantity demanded.

q represents the output quantity produced.

q^* represents the output quantity of beginning inventory.

q^d represents the output quantity that it is desired to have on hand at the beginning of the period.

q^s represents the output quantity shipped.

q^u represents the output quantity of unfilled demand.

p represents the output price.

There are k + h + m inputs of which the first k are labor, the next h are types of machines, and the remaining m are materials. Because some of them are complicated, the elements of some of the input vectors are shown in Appendix A. x^L and v are vectors of nk elements.

x^L represents the input quantity of labor used.

v represents the labor price.

For $i = 1,\ldots,k$ and $j = 1,\ldots,n$

$$x^L = (x_i^{Lj})$$

$$v = (v_i^j).$$

It should not be surprising that the cost to be allocated to the inputs is permitted to vary according to which product the input is applied. If this is not the case, the vectors v, y, and r will be much simpler. The first k elements of x^L and v pertain to product one, the next k elements pertain to product two, etc. (see Appendix A). The total amount of labor type i that is used on all products will be denoted by X_i^L where

$$X_i^L = x_i^{L1} + x_i^{L2} + \ldots + x_i^{Ln} \qquad \text{for } i=1,\ldots,k.$$

X^L will represent a k by 1 vector of the total amounts of the k types of labor used.

x^M and y have nh elements.

x^M represents the input quantity of machine hours used.

y represents the cost assessed to the use of machinery.

For i = 1,...,h and j = 1,...,n

$$x^M = (x_i^{Mj})$$

$$y = (y_i^j).$$

The first h elements of these two vectors pertain to product one, the next h to product two, etc. (see Appendix A). The total number of machine hours used for the i th type of machine is X_i^M.

$$X_i^M = x_i^{M1} + x_i^{M2} + \ldots + x_i^{Mn} \qquad \text{for } i=1,\ldots,h.$$

X^M will represent a column vector of h elements of these X_i^M 's.

$\bar{x}$ and r are column vectors of nm elements each.

$\bar{x}$ represents the material input quantity used.

r represents the material input price.

The first m elements of $\bar{x}$ and r are relevant to product one, the second m elements are relevant to product two, etc. (see Appendix A). For $i = 1,\ldots,m$ and $j = 1,\ldots,n$.

$$\bar{x} = (\bar{x}_i^j)$$

$$r = (r_i^j)$$

Analogous to X_i^L and X_i^M, the total amount of the i th input used among all n products is $\bar{X}_i$, where

$$\bar{X}_i = \bar{x}_i^1 + \bar{x}_i^2 + \ldots + \bar{x}_i^n \qquad \text{for } i=1,\ldots,m$$

and $\bar{X}$ is the vector having the elements $\bar{X}_i$.

Most of the other variables associated with materials will be m by 1 because no distinction needs to be made as to their allocation among products. X, X^d, X^*, and X^p are column vectors of m elements.

X represents the material input quantity received during the period.

X^d represents the desired level at which the material input is to be maintained at the end of the period.

X^* represents the material input quantity of beginning inventory.

X^p represents the material input quantity ordered.

To make the dynamic properties of the model more evident, discrete time periods will be assumed for the remainder of this chapter. All of the other variables are assumed to be continuous.

B. The Production Function

The production function to be used in this study is the simple linear one.

$$q = a + A^L x^L + A^M x^M + \bar{A}\,\bar{x} \qquad \text{(Eq. 2-1)}$$

where

a is an n by 1 vector of constants,

A^L is an n by nk coefficient matrix for labor,

A^M is an n by nh coefficient matrix for machinery, and

$\bar{A}$ is an n by nm coefficient matrix for materials.

The intercept, a, represents the combined fixed factors of production. This function is not applicable to long run problems unless a is defined to be zero and all of the inputs are assumed to be variable. The elements of the matrices A^L, A^M, and $\bar{A}$ are the marginal products of the respective inputs for each of the n outputs. The properties of this function have been described at great length in many micro-economic theory textbooks. One of the more lucid discussions can be found in Henderson and Quandt (41) where the linear homogeneous production of the first degree is delineated (a=0). The sizes of these coefficient matrices may seem unusual, but note that within each row the only non-zero elements are those which relate the relevant set of inputs to the output. A^L, A^M, and $\bar{A}$ are block diagonal matrices with blocks of 1 by k, 1 by h, and 1 by m, respectively. For instance, in the first row of A^L only the first k elements may be non-zero. For the second row of A^L the first k elements are zeroes, the next k elements may be non-zero, and all the remaining elements are zero. A^M and $\bar{A}$ have similar features (see Appendix A).

The rationale for selecting this linear production function as opposed to the Cobb-Douglas or a nonlinear function has two bases. First, recall that the purpose of this study is to formulate a mathematical programming model to describe the firm's decision processes identified in Figure 1. The programming problems to be constructed require that the constraint set be linear. Therefore, it would be impossible to include a nonlinear or

Cobb-Douglas production function in the final model. (Piecewise linear functions could usually be substituted for most nonlinear cases.) Second, the decisions to be derived from the model are not to be long run actions. This must be the case since capital expansion will hardly be mentioned in this study.

The restrictiveness of a linear production function for the short run is not so severe as it may seem. Ijiri (48, p. 15) has shown how easily a piecewise linear function can be fitted to represent a nonlinear path, however ugly it may appear to be. Although the Cobb-Douglas production function is often applied to industry and macro-economic studies, this does not preclude the assumption that individual firms have linear production functions. Houthakker (47) has shown this.

An excellent theoretical and empirical survey of production functions and their implied cost curves has been assembled by A. A. Walters (93). Mentioned in that article are most of the major studies of production functions in agriculture, business, and micro- and macro-economic theory.

C. The Input Prices

The input prices are the per unit average costs that are associated with the production process. Some of these are incurred independently of the level of utilization while others depend on the number of units of inputs that are employed. Therefore, the input price function will consist of a constant plus a function of the input. These are the firm's demand functions for the inputs. Suppose that

$$v = g^L + v(x^L)$$

$$y = g^M + y(x^M)$$

$$r = g + r(x)$$

All three of these will be assumed to be representable by linear functions. Again piecewise functions could be used if necessary. The properties of these functions will be given separately.

The average cost of employing the labor force is assumed to be

$$v = g^L + G^L x^L \qquad \text{(Eq. 2-2)}$$

where g^L is an nk by 1 vector and G^L is an nk by nk matrix. The off diagonal elements of the coefficient matrix reflect substitutability and complementarity of the types of labor among themselves in the production of the various products. The elements along the main diagonal of G^L represent the change in v for a change in the respective element of x^L; whether these values will be positive or negative cannot be known in general. Costs of training and the chance of crowding workers cause the main diagonal elements of G^L to be positive while the increased efficiency from additional specialization of tasks influences these elements towards being negative. These costs may also vary for one type of labor when it is applied to various products.

When a unit of machinery is purchased, the method and rate of depreciation are determined by the firm. This shall be reflected in an nh by 1 vector of constants, g^M. However, wear beyond normal use and obsolescence may cause a machine to be disposed of or have its use cutback before it is fully depreciated. In any particular period one of these additional burdens will be reflected in a positive value along the main diagonal of G^M which is an nh by nh matrix. A negative value in one of these positions will reflect increased efficiency of utilization of the machine in question. The off diagonal elements of G^M indicate the possibility of using one machine in place of another. The average costs of utilizing the machinery are given by

$$y = g^M + G^M x^M . \qquad \text{(Eq. 2-3)}$$

For the materials that are purchased the situation is analogous to that of labor and machinery. Let

$$r = g + G x \qquad \text{(Eq. 2-4)}$$

where g is a column vector of nm constants and G is an nm by nm matrix. The off diagonal elements of G will reflect the complementarity and substitutability of the types of materials among themselves in the production of each product.

This simple linear demand function is the case when materials might be purchased in imperfectly competitive markets. The implications of purchasing the materials in a market with constant prices can be seen by defining G to be a null matrix. According to the accepted economic theory, the elements on the main diagonal of G will rarely be positive since these are the changes in the input prices for the respective change in the quantity purchased. The off diagonal elements of G should help to reflect whether the various materials can be substituted for one another in the production of each product. Allowing the independent variable to be x instead of X includes the possibility that two materials may be substitutes in the production of one product and complements in the production of a second item. Besides, there is no reason to expect each material to be utilized in the production of each item. Most of the costs of ordering, shipping, and receiving the materials are included in g. However, some of the average costs decline as the quantity increases because the shipments can be handled more efficiently. Also, there is the possibility of obtaining quantity discounts for large orders and convincing the suppliers to pay the shipping charges.

Equations 2-2, 2-3, and 2-4 may have to be replaced by piecewise linear functions if the slope of each is not approximately constant. The elements of the matrices G^L, G^M, and G reflect only the substitutability or complementarity among materials themselves (elements of G), among types of labor themselves (elements of G^L), and among the machinery types (elements of G^M) when applied to the various products. The substitutability or complementarity among labor, machinery, and materials is not reflected in these matrices. Relations among the classes of inputs (materials, labor, and machinery) for each product is the role of the isoquants determined by Equation 2-1 after a level of production is fixed for a particular item.

There are numerous other costs to be accounted for and controlled by this model. These will be discussed in chapter three along with the objective functions to be studied.

D. Varying the Level of Production

The decision to change the level of production between periods t and t+1 is assumed to be based on the level of output inventory, q^*_{t+1}, unfilled demand, q^u_t, and the desired level of beginning inventory for the t+1 st period, q^d_{t+1}. Within the model q^u_t can be treated as a negative inventory. However, the costs of having a unit of unfilled demand are vastly different from those of having one unit of inventory on hand. This will be shown in detail in chapter three. The amount available for sale in period t is the beginning inventory, q^*_t, plus the output produced in the period, q_t. If the amount available is at least as large as the quantity demanded in period t, $q^u_t = 0$ and $q^*_{t+1} \geq 0$. Otherwise $q^*_{t+1} = 0$ and $q^u_t > 0$.

Let

$$W_t = q_t^* + q_t - \overline{q}_t \qquad \text{(Eq. 2-5)}$$

Then if $q_t^* + q_t \geq \overline{q}_t$, $W_t = q_{t+1}^* \geq 0$; if $q_t^* + q_t < \overline{q}_t$, $W_t = -q_t^u < 0$.

The value of W_t can be expected to be a major factor in determining how production will change from period t to t+1. One should expect $q_{t+1} < q_t$ if $W_t > q_{t+1}^d$, since this implies that $q_t^u = 0$ and $q_{t+1}^* > 0$. Likewise, $q_{t+1} > q_t$ if $W_t < q_{t+1}^d$, since that can occur only when $q_{t+1}^* = 0$ and $q_t^u > 0$. In general $q_{t+1} - q_t$ can be expected to adapt to the value of W_t. Of equal or greater importance are the cumulations and changes of W over time. If $\sum_{j=0}^{N} W_{t-j}$ is a large positive number, the level of inventory being stored is very great. Many units are probably being stored for several periods. In this case a production reduction is in order. If $\sum_{j=0}^{N} W_{t-j}$ is a large negative number, back orders or unfilled demands are occurring continuously. In either case production should be increased. Also to be considered is the change in W over time: $W_t - W_{t-1}$. Throughout this study the difference operator, Δ, will be used to mean the backward difference; therefore, $\Delta W \equiv W_t - W_{t-1}$. This difference will account for whether or not the production level was more adequate in period t than in the previous period. Assuming K_p, K_d, and K_c to be n by n matrices and K_0 to be an n by one vector of known constants, the change in the level of output will be determined by Equation 2-6. In order to consider the most general case, suppose that it is desired to maintain q_t^d units of inventory in the t th period. Then when $W_t = q_{t+1}^d$, the process is working as well as is possible. The constants K_0 are included to allow for the level of production to change over time. Therefore,

$$q_{t+1} - q_t = K_0 + K_p(W_t - q^d_{t+1}) + K_d \Delta (W_t - q^d_{t+1}) +$$

$$K_c \sum_{j=0}^{N} (W_{t-j} - q^d_{t-j+1}) \qquad \text{(Eq. 2-6)}$$

where t=0 is the initial and t=N is the earliest time period that is permitted to affect the output decision for the t+1 st period.

One of the more significant applications of a process of this sort was completed by A. W. Phillips (70). In his study Phillips assumes continuous time periods and identifies three components which he called the proportional, derivative, and integral policies. The proportional policy is represented by $K_p(W_t - q^d_{t+1})$ in Equation 2-6. The derivative element is $K_d \Delta (W_t - q^d_{t+1})$. The integral policy is $K_c \sum_{j=0}^{N} (W_{t-j} - q^d_{t-j})$. In Phillips' study the purpose was to select a formula that would be applicable to stabilizing a macro-economic model, but the implications and roles of the three elements for this problem are the same. Phillips' shows that the elements along the main diagonal of K_p, K_d, and K_c must be negative to stabilize the system. When applied alone the proportional policy has two major shortcomings. First, it does not guarantee correction for the whole deviation. Second it promotes a cyclical fluctuation in the time path for $W_t - q^d_{t+1}$. Phillips argues (70) that most of the trend and cyclical effects can be eliminated by adding the derivative and integral policies to the proportional one.

Adaptive relationships such as Equation 2-6 have been prominent in economics for many years. As Mills points out (59), such functions abound in the literature of economics since the 1930's. The Swedish economists concentrated on relating expected and actual events. One of the early applications to inventory theory was Metzler's study of stability in inventory cycles (58). Griffith Evans suggested that the response of the quantity demanded to small price changes be studied using a relationship that is analogous to Equation 2-6 (27). From Metzler's and Evans' studies one can see that an adaptive relationship is almost always being analyzed when distributed lag models are studied. The adjustment processes of excess demand and supply equations can also be put into such a framework (41).

Mills' (59) and Phillips' (70) works show the latitude to which such relationships might be applied. Murphy (61) and Sworder (86) are studying the application of similar processes to economic policy and statistical games. Their suggestions on dealing with incomplete and the lack of information can be expected to play a significant role in some future work in both micro- and macro-dynamics. In studies where dynamic optimization problems are to be solved, relationships such as Equation 2-6 commonly describe part of the constraint set. Examples may be found in both theoretical (17) and applied (79) articles.

E. The Purchasing Decision

The purchasing of raw materials involves two considerations: one is the lead time and the second is the quantity to be ordered. The lead time is the number of periods that pass between ordering and receiving materials. If the lead time is known to be exactly T periods,

$$X_t = X^P_{t-T}$$

where the total quantity received in period t is the vector X_t and the total quantity ordered T periods before t is X^P_{t-T}. (These variables were defined in the introduction to this chapter.) Assuming the lead times to be known exactly is hardly realistic, but it is not unlikely that the means and variances of the lead times can be estimated.

Suppose that b is an m by 1 vector of continuous stationary random variables with vector bounds b_L and b_U so that $b_L \leq b \leq b_U$. Associated with each element of b is a known or estimated probability density function $f_b^i(\cdot)$; $f_b(\cdot)$ is an m by 1 vector of these density functions. Let $f_b^i(b_i^*)$ be the probability that the difference between the quantities ordered in period t-T and received in period t is b_i^* for the i th material. If b_i^* is positive, more is received in period t than was ordered in period t-T. If b_i^* is negative, either the shipment was late or the lead time was insufficient. Each situation may create a problem. If the shipment is late, there may be an unfilled need for the material on the production line. If the material arrives too early, it may require the temporary acquisition of an additional storage facility. A more reasonable description of the lead time and ordering problem than was given in the previous paragraph would be

$$X_t = X_{t-T}^p + b \qquad \text{(Eq. 2-7)}$$

in which stochastic arrival errors or miscalculations are permitted.

The firm's learning and adaptations on the quantities to be ordered in period t, X_t^p, is formulated in a similar manner to that of the change in production in the previous section. There is the possibility of having too much of a material on hand and incurring a storage problem. Likewise, a material shortage could cause a production slow down, require rush orders on purchases, and cause much of the labor force to be idle. Letting X^d be the vector of target levels of materials inventory to be maintained, the firm should choose its receipts of materials in period t to respond to the level of $X^* - X^d$ at the end of the t-1 st period. If immediate delivery of materials is possible (T = 0), this can be done exactly; however, it is more likely that a lead time is required. For materials received in period t, X_t, to account for any $X^* \neq X^d$, this difference must be considered in period t-T. This is because period t is the soonest that the deviation occurring in period t-T can be adjusted. Therefore,

$$X^p_{t-T} = H_0 + H_p(X^*_{t-T} - X^d_{t-T}) + H_d\Delta\,(X^*_{t-T} - X^d_{t-T}) + H_c \sum_{j=0}^{M}(X^*_{t-T-j} - X^d_{t-T-J}) \qquad \text{(Eq. 2-8)}$$

where M is the earliest time that affects X^p_{t-T}. H_0 is an m by 1 vector of constants to allow for a normal order that is not a response to the difference between the inventory and target levels. H_p, H_d, and H_c are m by m matrices of parameters that are analogous to K_p, K_d, and K_c respectively. The role of this response function for materials is analogous to that of Equation 2-6 for the change in the level of production. Both involve the same types of components which are proportional, derivative, and integral, and both are to stabilize decision making. The main diagonals of H_p, H_d, and H_c must have negative elements as was the case for K_p, K_d, and K_c.

Finally, the bookkeeping device for materials inventory must be included. This is simply that inventory will change from one period to the next by the difference between the amount received and the amount used in the period. Therefore,

$$X^*_{t+1} = X^*_t + X_t - \overline{X}_t. \qquad \text{(Eq. 2-9)}$$

F. The Demand for the Output

The demand and revenue structure is where the stochastic element becomes most crucial in the model. The quantity demanded, $\overline{q}$, is assumed to be a non-negative continuous n by 1 vector of stationary random variables having a vector of probability density functions $f_{\overline{q}}(\cdot)$. Then,

$\overline{q}$ is distributed as $f_{\overline{q}}(\cdot)$.

The specific demand distributions are to be derived or estimated from the data. These will be considered in chapter five where an application of the model is to be attempted.

The output price will be a linear (piecewise, if necessary) function of the quantity demanded,

$$p = s + S\,\overline{q} \qquad \text{(Eq. 2-10)}$$

where s is an n by 1 column vector of intercepts for the demand function and S is an n by n matrix of slopes. The off diagonal elements of S will show the interrelationships among the outputs. This demand function allows the firm to vary the price to respond to the quantity demanded. In economic theory it is usually assumed that the elements of s are non-negative and that the elements along the main diagonal of S are not positive.

The firm's total revenue is the scalar product of the vectors of prices and quantities actually shipped to purchasers.

$$TR = q^{s}\,p = q^{s\prime}\,s + q^{s\prime}\,S\,\overline{q} \qquad \text{(Eq. 2-11)}$$

When no distinction is made between q^s and $\overline{q}$ (quantity shipped always equals the quantity demanded), $TR = \overline{q}\,s + \overline{q}'\,S\,\overline{q}$; if all n of the $\overline{q}_i$'s are identically distributed random variables, and $q^s = \overline{q}$, the distribution of TR is the same as that for $\overline{q}_i$. This is a consequence of the fact that the distribution of $\overline{q}_i^2$ has been shown (63) to be a linear combination of $f_{\overline{q}_i}(\cdot)$.

$$f_{\overline{q}}(c) = \tfrac{1}{2}\sqrt{c}\left[f_{\overline{q}}(\sqrt{c}) + f_{\overline{q}}(-\sqrt{c})\right] \qquad c > 0$$

$$= 0 \qquad c < 0$$

where c is a particular value of $\overline{q}^2$.

One may be inclined to feel that this formulation of the demand structure removes the model from the context of mathematical economics and makes this a study in operations research. This is hardly the case since the interdependence of the output price and quantity demanded is not being ignored. Furthermore, the nature of the market structure for demand is a major factor in determining the total revenue function. For a perfectly competitive output market with a constant price, $p = s$ and $S = 0$, but $\overline{q}$'s being a stochastic random vector is not excluded.

Assuming the quantity demanded to be a random variable is common in many studies and surveys in mathematical economics. Among the better known are several chapters of Studies in the Mathematical Theory of Inventory and Production (2) and two survey articles on the inventory problem by Dvoretsky, Kiefer, and Wolfowitz (24, 25). Mills (59) has considered many of the implications of the demand and revenue structures described in this section. However, he does not extend his analysis to the implications for the production function and input side of the analysis. In the following section the model for this study is summarized for the convenience of the reader and for future reference.

G. Summary

In summary, the model to be considered in this study is:

$$q = a + A^L x^L + A^M x^M + \overline{A}\,\overline{x} \qquad \text{(Eq. 2-1)}$$

$$v = g^L + G^L x^L \qquad \text{(Eq. 2-2)}$$

$$y = g^M + G^M x^M \qquad \text{(Eq. 2-3)}$$

$$r = g + G\,x \tag{Eq. 2-4}$$

$$W_t = q_t^* + q_t - \overline{q}_t \tag{Eq. 2-5}$$

$$q_{t+1} - q_t = K_0 + K_p(W_t - q_{t+1}^d) + K_d\Delta\,(W_t - q_{t+1}^d) + K_c \sum_{j=0}^{N} (W_{t-j} - q_{t-j+1}^d) \tag{Eq. 2-6}$$

$$X_t = X_{t-T}^p + b\ ,\ b \sim f_b(\cdot)\ ,\ b_L \leq b \leq b_U \tag{Eq. 2-7}$$

$$X_{t-T}^p = H_0 + H_p(X_{t-T}^* - X_{t-T}^d) + H_d\Delta\,(X_{t-T}^* - X_{t-T}^d) + H_c \sum_{j=0}^{M} (X_{t-T-j}^* - X_{t-T-j}^d) \tag{Eq. 2-8}$$

$$X_{t+1}^* = X_t^* + X_t - \overline{X}_t \tag{Eq. 2-9}$$

$$X = x^1 + x^2 + \ldots + x^n$$

$$\overline{X} = \overline{x}^1 + \overline{x}^2 + \ldots + \overline{x}^n$$

$$p = s + S\,\overline{q}\ ,\ \overline{q} \sim f_{\overline{q}}(\cdot)\ ,\ \overline{q} \geq 0 \tag{Eq. 2-10}$$

$$TR = q^{s\prime}\,p = q^{s\prime}\,s + q^{s\prime}\,S\,\overline{q} \tag{Eq. 2-11}$$

That each of these equations represents a set of functions should not be forgotten. Recall, that appropriate vectors and matrices have been defined for all of the variables and parameters so that a multi-product, multi-input firm is represented by the model. The role of the off diagonal elements has been mentioned whenever there was a clear interpretation.

In the future chapters Equations 2-1 through 2-11 will form the constraints for an optimization problem. Quadratic programming problems with stochastic variables will be constructed. For each of these it is important that the constraints be linear. This is the motivation for assuming piecewise linearity although nonlinear functions might give as good or better representations for some of the functions.

III. THE OBJECTIVE FUNCTIONS

A. Introduction

The costs that were considered in chapter two as a part of the basic model are those that must be incurred if the firm is to remain in operation. It has been presumed that the firm uses labor, materials, and machinery as its variable inputs to produce its various products. The other costs and losses to be accounted for in this study could conceivably be avoided while the firm remains in business. Incurring or avoiding these costs will depend on the efficiency and attitude of the firm's management. For instance, it is usually preferred to incur small costs for storing materials and finished goods rather than to risk a production slow down or unfilled customer demand. It is also common to overestimate lead times for similar protection.

In this chapter the main emphasis will be on the construction of the total inventory cost function, TIC. These plus the costs of materials, labor, and machinery will represent the total variable costs, TVC, of the firm. The fixed costs will not be specifically included here since the firm's short run decisions cannot influence them. The total variable costs, TVC, will be used to formulate the objective functions which will be proposed as the possible representations of the purposes for which the firm operates.

B. The Total Variable Cost Function

The total production costs, TPC, will be the sum of the expenditures on labor, materials, and machinery. These expenditures are determined from the average cost functions delineated in Equations 2-2, 2-3, and 2-4.

$$TPC = x^{L\,'} g^{L} + x^{L\,'} G^{L} x^{L} + x^{M\,'} g^{M} + x^{M'} G^{M} x^{M} +$$
$$x' g + x' G x \qquad \text{(Eq. 3-1)}$$

Many studies emphasize the costs associated with changes in the level of production. Mills (59), Holt and his associates (46), and several articles in Studies in the Mathematical Theory of Inventory and Production (2) are included in this group. This is implicitly in the model delineated in chapter two since a production function and production costs are specified. An increase in q will imply increases in x, x^{M}, and x^{L} which in turn cause TPC to change.

The three studies cited in the previous paragraph all consider the costs of holding an inventory of finished goods. All include the losses involved in having unfilled demand. In addition to penalties incurred for maintaining output and not filling customer demands, two other assessments will be allowed in this study. One is the cost of holding materials inventory; the other is the loss incurred by running out of materials. In the later case, the penalty may range from that of a rush order to the curtailing of production.

Closest to the cost function to be studied here is that analyzed by Holt, Modigliani, Muth, and Simon (46). That study has been used by Theil (89, Ch. 3) to exemplify his work or decision making with quadratic objective functions. In nearly all of the studies that are relevant to this one, costs are not discounted over time. This will be the case here, also. Since this study deals with short run decision making, such an assessment would hardly be useful.

The cost of holding inventory of materials in any period is the average cost of holding one unit of the material in question times the number of units held throughout the period. The quantity held throughout the period can be measured by the change in the number of units on hand from one period to the next. Suppose that $\emptyset_i$ cubic feet are required to store one unit of material i for one period and that the average cost allocated to $\emptyset_i$ feet of warehouse space is c_i^*. Then c^* will be a column vector of these costs for each of the m materials. An assessment is to be made only if a unit of inventory is held throughout the full period. The penalty or cost for holding materials inventory in the t th period will be

$$c^{*'} (X_{t+1}^* - X_t^*) = c^{*'} \Delta X^*.$$

If it is always desired to hold $X^d \neq 0$ units of inventory, there is a cost to be incurred for doing so. This cost is decided upon by the firm in order to avoid the possibility of a more serious loss from not having enough materials to maintain the desired level of production.

From Equation 2-5 and the line that follows it in chapter two, it was shown that finished goods and unfilled demand can be analyzed via one relationship. The assessments for $W_t = q_{t+1}^*$ or $W_t = -q_t^u$ are determined while assuming the cost of these cases to be a quadratic function. Let the cost of $W_t \neq 0$ be defined by

$$W_t' \bar{c} W_t - \hat{c}' W_t$$

where $\bar{c}$ is an n by n diagonal matrix and $\hat{c}$ is an n by 1 column vector. All of the non-zero coefficients of this function are defined to be positive. For the i th product or output, the assessment is

$$\bar{c}_{ii} W_{it}^2 - \hat{c}_i W_{it} .$$

The two parameters to be specified are $\bar{c}_{ii}$ and $\hat{c}_i$.

Let c_i be the known average cost of holding one unit of finished goods inventory. When $W_t \geq 0$, then $q^*_{t+1} \geq 0$ and $q^u_t = 0$. $c_i\, q^*_{it+1}$ is the total cost of holding q^*_{it+1} units at the beginning of period t+1 and, therefore, at the end of period t.

$$\bar{c}_{ii}\ (q^*_{it+1})^2 - \hat{c}_i\ (q^*_{it+1}) = c_i\ q^*_{it+1}$$

Let c^u_i be the given average loss from having one unit of unfilled demand of output i. This loss includes the price that would have been received for the product plus the sum of per unit future profits that may be lost if the customer finds a new source for his needs. For the t th period when there is unfilled demand, $W_t = -q^u_t$ and $q^*_{t+1} = 0$; $c^u_i\, q^u_{it}$ is the total loss from having q^u_{it} units of unfilled demand in period t.

$$\bar{c}_{ii}\ (-q^u_{it})^2 - \hat{c}_i\ (-q^u_{it}) = c^u_i\ q^u_{it}$$

Solving the two relations of this paragraph for the unspecified coefficients, namely $\bar{c}_{ii}$ and $\hat{c}_i$, gives

$$\bar{c}_{ii} = \frac{c_i + c^u_i}{q^*_{it+1} + q^u_{it}} \quad \text{and} \quad \hat{c}_i = \frac{c^u_i\, q^*_{it+1} - c_i\, q^u_{it}}{q^*_{it+1} + q^u_{it}}$$

Given the cost of q^u_{it} units of unfilled demand to be c^u_i and the cost of holding q^*_{it+1} units of inventory to be c_i, $\bar{c}_{ii}$ and $\hat{c}_i$ are determined for the relevant levels of q^*_{it+1} and q^u_{it}.

Having the cost coefficients be functions of the level of inventory and unfilled demand is not accidental. There is rarely enough information for a statistical analysis to estimate $\bar{c}_{ii}$ and $\hat{c}_i$. Furthermore, values for c^u_i and c_i are very much in the minds of the firm's decision makers when they elect to maintain an inventory or risk having some unfilled demand.

The values that are selected for c_i and c_i^u by the firm are very much related to, if not determined by, the levels of q_{it+1}^* and q_{it}^u that are believed to be likely. Holt and his associates (46, p. 74) determined some of the cost coefficients in their study by solving equations where values from actual occurrences were substituted for unknowns.

It is surely to be expected that c_i^u will be larger than c_i. For k units of q_{it+1}^* versus the same number of units of q_{it}^u,

$$c_i \; k = \bar{c}_{ii} \; (k^2) - \hat{c}_i \; (k)$$

and

$$c_i^u \; k = \bar{c}_{ii} \; (-k)^2 - \hat{c}_i \; (-k) .$$

Subtracting the second from the first of these expressions leaves

$$c_i^u \; k - c_i \; k = 2 \; \hat{c}_i \; k$$

and

$$c_i^u - c_i = 2 \; \hat{c}_i > 0 \; .$$

The total costs for inventory control and unfilled demand is defined to be TIC, where

$$\mathrm{TIC} = W' \; \bar{c} \; W - \hat{c}' \; W + c^* \; \Delta \; X^* . \qquad \text{(Eq. 3-2)}$$

Note that while a cost was assessed only for a change in the level of materials inventory, a loss is incurred for any output inventory that is on hand at the beginning of a period. This is because there is capital tied up in any finished good that is not sold immediately; this capital represents that value of all of the inputs utilized in the production of the finished good. The total variable costs, TVC, to be considered in this study are:

$$TVC = TPC + TIC = x'\,g + x'\,G\,x + x^{L}{}'\,g^{L} + x^{L}{}'\,G^{L}x^{L} + x^{M}{}'\,g^{M} + x^{M}{}'\,G^{M}x^{M} + W'\,\overline{c}\,W - \hat{c}'\,W + c^{*}{}'\,\Delta X^{*}. \qquad \text{(Eq. 3-3)}$$

The firm's profit, P, under these circumstances is given by

$$P = TR - TVC - TFC \qquad \text{(Eq. 3-4)}$$

where TR and TVC are defined by Equations 2-11 and 3-3, respectively, and TFC represents the fixed costs incurred by the firm. The fixed costs will not be of concern here because any result that applies to TR - TVC will apply to P in the short run.

Some of the firm's inputs are likely to be purchased for fixed prices, regardless of the quantity demanded. For the materials, labor, or machinery if the average cost is constant, the average cost equals the marginal cost, and this will be the price paid or cost assessed. For this to be represented within the model of chapter two, all of the elements of the relevant row of G^L, G^M or G will be zeroes. However, the total variable cost function (Equation 3-3) is still a quadratic form. In fact, unless Equations 2-5 and 2-6 are revised so that $q^* + q > \overline{q}$ is handled in a very different way from $q^* + q < \overline{q}$, it is unlikely that a sensible linear cost function can be constructed for the variable costs to be considered here.

That a quadratic function is convex is a mathematical consideration that should not be ignored. This feature of the cost function under consideration means that Equation 3-3 may serve as an objective function to be minimized subject to the model delineated in chapter two. Problems of that sort have been studied for nearly twenty years. Three of the early efforts on quadratic programming problems were by Markowitz (57), Simon (83), and Frank and Wolfe (35). More recently Theil (89) and Holt and his associates (46) have solved applied problems. The studies on programming with random variables will be mentioned in chapter four. The solutions to the particular problems to be solved in this study will require the application of the methods of quadratic programming.

C. Programming Problems Relevant to this Study

Two programming problems will be viewed in this study. These will represent only a few of the multitude of possible problems that might be examined with the model of chapter two as the basis for the constraint set. The particular problems to be solved are:

(1) maximizing the firm's profits, P, allowing the input and output prices to be linear functions of their respective quantities; and

(2) minimizing the firm's total variables costs, TVC, under the same price assumptions as for problem (1).

Each of these problems will have a quadratic objective function with linear constraints which are represented by the model in chapter two. Problems one and two require that all of the elements of the matrices G^L, G^M, G, and S be tested to see if they are zero. Assuming that the price of an input or output is constant usually implies that the particular market is perfectly competitive. While a fixed price may be a necessary condition for perfect competition, this is not a sufficient condition.

Fixed prices are probably more common than not in short run agreements. It is likely that the prices to be paid for labor and materials and received for finished goods are determined to be constants by contract. The best known example might be the agreements by which labor is paid; for the short run the wages are most likely fixed with the condition that adjustments be made regularly. These agreements do not suggest that there is perfect knowledge about the labor market, nor that there are unlimited quantities available, nor that the inputs are homogeneous within their class.

There are numerous other optimization problems that one may want to study in this context. Baumol's (3) suggestion that sales maximization is more realistic than profit maximization can be considered by maximizing TR; for that problem a constraint on profits would need to be added to the model.

A budget constraint can also be added to restrain the firm's variable costs for the firm's expenditure on materials, labor, and machinery. One possibility would be to require that

$$TPC + TIC \gtrless B_t$$

where B_t is the planned expenditure or budget for the period. Then the slack variable in the programming problem can be interpreted to be the amount that is borrowed if $TPC + TIC > B_t$ or the amount that is lent if $TPC + TIC < B_t$. This can be put into a programming problem by using two slack variables y^+ and y^- where $TPC + TIC + y^+ - y^- = B_t$ and either y^+ or y^- or both are zero. None of the problems described in this paragraph will be tackled in this study. However, these possibilities could all be analyzed by minor variations in the constraint set formed by Equations 2-1 through 2-10; note also that the problems would still be within the confines of applications of quadratic programming.

All of the problems in which Equations 2-1 and 2-7 are among the constraint set involve n + m random variables $(\bar{q}_1, \ldots, \bar{q}_n, b_1, \ldots, b_m)$. This feature of the model has hardly been emphasized in this chapter for this will be the main consideration of chapter four. In that chapter the various methods for handling programming under uncertainty will be summarized.

D. The Firm's Goals in the Programming Framework

Whether a firm's motives for operating can be summarized by one or more equations is questionable. Martin Shubik's (80) study of the goals to which firms attested indicates that many goals cannot be quantified. Also there is the question of whether a firm's multiple goals can be summarized by one goal such as profit maximization or minimizing costs.

For this study it shall be assumed that an objective function can provide a reasonable representation of the firm's preferences and goals. By selecting policies that optimize the preference function subject to the model of the firm, the organization is better off than if this knowledge were being ignored. This does not imply that situations not representable by equations are to be ignored.

The objective functions that were proposed in the previous section of this chapter shall be defined to represent multiple goaled considerations. Each variable in the objective function represents one goal. In this analysis Ijiri's argument is being accepted (48; Ch. 3). He suggests that optimizing z where $z = \sum_{j=1}^{n} a_j x_j$ is to consider n subgoals--$x_1, \ldots, x_n$--and one aggregate goal--z. The a_j's are the relevant weights given to the n subgoals. In this sense any objective function having more than one variable has multiple subgoals.

Ijiri continues by suggesting that in order to study p goals, p functions are needed. For example p goals could be specified by

$$z_1 = \sum_{j=1}^{n} a_{1j}\, x_j$$

$$z_2 = \sum_{j=1}^{n} a_{2j}\, x_j$$

$$\vdots$$

$$z_p = \sum_{j=1}^{n} a_{pj}\, x_j$$

Extending this analysis, a function having multiple goals would such as

$$Z = \sum_{i=1}^{p} b_i\, z_i = \sum_{i=1}^{p} \sum_{j=1}^{n} b_i\, a_{ij}\, x_{ij}$$

where the previous p equalities must also hold. Therefore, by studying Z along with the p equations, multiple goals are being considered. Note that Z can be a scalar objective such as total revenue, total cost, total variable cost, or profits to be optimized with each of the p equalities being a member of the constraint set.

Therefore, the problems that have been outlined for this study will be defined to be cases where a multi-purpose firm is being analyzed. That the optimal value for the objective function is a scalar is true, but the values of the variables that determine this scalar will also determine the specific equalities of the model.

IV. A SURVEY OF OPTIMIZATION TECHNIQUES

A. Introduction

The difficulties of dealing with risk and uncertainty in programming problems have been discussed in the literature for nearly fifteen years. Among the early investigators were Freund (36), Simon (83), Tintner (90), and Dantzig (19); Markowitz (56), Theil (88), and Charnes and Cooper (8) followed shortly. Recently Sengupta (77) has added several theorems to the literature on linear stochastic programming.

As outlined in the previous chapters, the problems to be solved in this study require that a quadratic objective function be optimized subject to a set of linear constraints. Because of the uncertainty in material arrivals, b_1, b_2, ..., b_m are random variables; the unknown output quantities demanded, $\bar{q}_1$, $\bar{q}_2$, ..., $\bar{q}_n$, are also random variables. Therefore, $n + m$ random variables would be included in the quadratic programming problem in each time period. Throughout most of this chapter such problems will be discussed in general so that several of the many methods for handling risk and uncertainty in optimization problems can be compared.

One feature of all of the methods to be reviewed in this chapter should be pointed out. Each of these techniques requires that the probability density function of every random variable be assumed or estimated. Therefore, all of the methods for handling random variables that are described here are techniques for studying programming under risk. The terms risk and uncertainty will be used interchangeably here as is often done in the relevant literature.

The most common method for dealing with uncertainty in programming problems is certainty equivalence; this technique is described in section B and related to the common practice of optimizing the expected value of objective functions that involve random variables. Among the other methods of programming under risk and uncertainty that will be described are: the passive approach to stochastic programming, the active approach to stochastic programming, chance-constrained programming, and multi-stage programming. The role that time plays in these problems will be discussed in section G.

B. Uncertainty and Certainty Equivalence

Madansky (55) summarizes the usual methods for handling uncertainty to be

(1) the replacement of random elements by their expected values,

(2) the replacement of random elements by the most pessimistic estimate for each, or

(3) the recasting of the problem into a two-stage problem in which the second stage compensates for the inaccuracies of the first.

The most common practice seems to be the first of these. Simon studied this method of handling random variables in a quadratic programming problem (83). Theil has extended this work (89); in his notation the quadratic programming problem is to:

$$\text{optimize } w(x,y) = a'x + b'y + \tfrac{1}{2}(x'Ax + y'By + x'Cy + y'C'x)$$

$$\text{subject to} \quad y = R\,x + s$$

where

a and x are m by 1 vectors

$$a = (a_j) \qquad x = (x_j)$$

b, y, and s are n by 1 vectors

$b = (b_i) \quad y = (y_i) \quad s = (s_i)$

R is an n by m matrix

$R = (r_{ij})$

A is an m by m symmetric matrix

$A = (a_{jh}) = (a_{hj}) = A'$

B is an n by n symmetric matrix

$B = (b_{ik}) = (b_{ki}) = B'$

C is an m by n matrix

$C = (c_{jk})$

$i, k = 1, \ldots, n$

$j, h = 1, \ldots, m$

Substituting Equation 3-2 into Equation 3-1 reduces $w(x,y)$ to be a function of only x, $w(x,Rx + s)$. One necessary condition for x^o to be an optimal vector of x's (where there are no random variables in the problem) is that $w(x,Rx + s)$ be differentiated with respect to x, and the result be set equal to zero and solved for x. The other necessary condition is that K be negative-definite to maximize w or positive-definite to minimize w. Together these necessary conditions are sufficient for $x = x^o$ to optimize Equation 3-1 subject to Equation 3-2. The results, which are proved by Theil (89) are:

$$w(x,Rx + s) = b's + \tfrac{1}{2}\, s'Bs + k'x + \tfrac{1}{2}\, x'Kx \qquad \text{(Eq. 3-3)}$$

$$K = A + R'BR + CR + R'C'$$

$$k' = a' + b'\, R + S'\, (BR + C')$$

$$k = a + R'b + (R'B' + C)s$$

$$x^o = - K^{-1}k \qquad \text{(Eq. 3-4)}$$

$$y^{o} = - RK^{-1}k + s \qquad \text{(Eq. 3-5)}$$

$$w^{o} = w(x^{o},Rx^{o} + s) = b's + \tfrac{1}{2} s'Bs - \tfrac{1}{2} x^{o\prime} Kx^{o} \qquad \text{(Eq. 3-6)}$$

or

$$w^{o} = w(x^{o},Rx^{o} + s) \; b's + \tfrac{1}{2} s'Bs - \tfrac{1}{2} k'K^{-1} k \qquad \text{(Eq. 3-7)}$$

where

x^{o} is the optimal level of x,

y^{o} is the optimal level of y, and

w^{o} is the optimal value of $w(x,Rx + s) = w(x,y)$

k is an m by 1 vector since: a is m by 1; R'b is an m by n times an n by 1; (R'B' + C) is an m by n times an n by n plus an m by n, which gives an m by n; since s is an n by 1, (R'B' + C)s is an m by n times an n by 1. Therefore, k is the sum of three m by 1 vectors, which gives an m by 1 vector.

K is a symmetric m by m matrix since: A is a symmetric m by m matrix; B, which is a symmetric n by n matrix, is premultiplied by R', an m by n matrix; CR + R'C' is the sum of an m by n times an n by m matrix and its transpose which gives a symmetric m by m matrix; and the sum of m by m symmetric matrices is a symmetric m by m matrix.

Since K is an m by m symmetric matrix K^{-1} has the same characteristics. For future reference K^{-1} shall be defined by

$$K^{-1} = \begin{bmatrix} K^{-1}_{11} & \cdots & K^{-1}_{1m} \\ \vdots & & \\ K^{-1}_{m1} & \cdots & K^{-1}_{mm} \end{bmatrix}$$

For most of these considerations Theil assumes that the elements of the matrices A, B, C, and R are known constants. When the elements of a, b, and s are assumed to be constants, the optimal solutions are as given by Equations 3-4, 3-5, 3-6, and 3-7. Note that the optimal levels of x and y (x^o and y^o, respectively) are linear combinations of a, b, and s. When these three vectors are assumed to be vectors of random variables, the expected value of x^o, denoted by $E(x^o)$, has the same form as x^o in Equation 3-4 except that the random variables in the solutions are replaced by their expected values. The same result holds for the expected value of y^o, $E(y^o)$.

$$E(x^o) = E(-K^{-1}\, k) = -K^{-1}\, E(k) \qquad \text{(Eq. 3-4a)}$$

$$E(y^o) = E(-RK^{-1}\, k + s) = -RK^{-1}\, E(k) + E(s) \qquad \text{(Eq. 3-5a)}$$

$$E(k) = E(a) + R'\, E(b) + (R'B' + C)\, E(s)$$

Besides these results, Theil (89) shows that

$$E(w^o) = E(b's) + \tfrac{1}{2}\, E(s'Bs) - \tfrac{1}{2}\, E(k'\, K^{-1}\, k) \qquad \text{(Eq. 3-7a)}$$

$$\begin{aligned} E(w^o) = {} & E(b')\, E(s) + \tfrac{1}{2}\, E(s')\, B\, E(s) - \tfrac{1}{2}\, E(k')\, K^{-1}\, E(k) + \\ & \left[E(b's) - E(b')\, E(s)\right] + \tfrac{1}{2}\left[E(s'Bs) - E(s')\, B\, E(s)\right] + \\ & \tfrac{1}{2}\left[E(k'\, K^{-1}\, k) - E(k')\, K^{-1}\, E(k)\right] \end{aligned}$$

or

$$\begin{aligned} E(w^o) = {} & E(b')\, E(s) + \tfrac{1}{2}\, E(s')\, B\, E(s) - \tfrac{1}{2}\, E(k')\, K^{-1}\, E(k) + \\ & COV(s'Bs) + \tfrac{1}{2}\, COV(k'\, K^{-1}\, k) \end{aligned}$$

Theil assumes constant variances and covariances. Therefore, the difference between $E[w(x^o, y^o)]$ and $w[E(x^o), E(y^o)]$ is a non-negative constant.

The importance of this result should not be overlooked. The decision to optimize the expected value of a quadratic objective function having random variables, namely $E[w(x,y)]$, provides the same set of optimal allocations $x = x^o$ and $y = y^o$, as the decision to optimize $w[E(x), E(y)]$. Optimizing objective functions with random variables by finding the optimal value of the expected value of that objective function is common. Among the most relevant to this study is that by Holt and his associates (46). It was this example that was reproduced by Theil in chapter five of Optimal Decision Rules for Government and Industry.

These are the main static results (89, pp. 52-58) that Theil derives for the certainty equivalent of the deterministic problem. Theil extends these results (in an appendix) to show the effect of permitting all of the elements of a, b, s, A, B, C, and R to be random variables (89, pp. 72-74). The difficulties become much greater, and the results less interesting. Besides considering the cases where all of the parameters are random variables, in the static problem, Theil describes the dynamic aspects of the problems (89, Ch. 4). To study these problems, the matrices and vectors of Equations 3-1 and 3-2 are time-partitioned. A new set of variables is introduced for each time period that is within the scope of the problem. This formulation will be delineated in section G of this chapter.

As has been indicated by Fox, Sengupta, and Thorbecke (34, Ch. 9), there are several other results and features to be considered along with the certainty equivalents mentioned in this section. One feature is the derivation of optimal levels for x and y when the random variables are replaced by other measures of their averages. The modal and median averages have been studied for the triangular distribution of the random variables in particular. When there is only one random variable to be considered and it has a symmetric unimodel distribution function, the mean, median, and mode will be the same.

A second feature of the method of certainty equivalents is that the information to be collected or the sample to be taken is minimal. The density function for the random variable need not be known or assumed. Only an estimate of the population mean is essential. The arithmetic mean of a set of observations from a random sample will provide a statistically unbiased estimate of the population mean. In this sense the solutions to the programming problem having random variables do not presume the knowledge or assumption of the random variable's density function.

Two difficulties of the method of certainty equivalents are very much related to the two features cited above. First, one might prefer that more than one estimate of a random variable be considered when decision strategies are conceived. Second, unless the mean and mode of a distribution are the same, even the most likely occurrence is not represented by the expected value. Therefore, particular attention must be given to the application of this method to problems with random variables whose probability density functions are skewed.

C. Passive Approach to Stochastic Programming

The first study of stochastic programming appeared in 1955 (90) when Tintner extended a linear programming study in agricultural economics. The method used by Tintner in that paper has since been labeled the passive approach to stochastic programming; Tintner, Sengupta, and their associates have used this title to distinguish the cases described in this section from those of section D, where the active approach to stochastic programming is discussed.

The passive approach to stochastic programming requires that the probability density function of $w(x,y)$ be derived. When the methods of mathematical statistics are not applicable, the distribution of $w(x,y)$ can usually be derived using numerical methods. Since $w(x,y) = w(x,Rx + s)$, it is sufficient to derive the distribution for Equation 3-3. It will be assumed that all of the random variables in the problem are independent random variables. Otherwise, as Tintner suggests (90, p. 227) apply orthogonal transformations to the random variables that are not independent; this will provide independent random variables on which to operate.

Enroute to deriving the density function for $w = w(x,Rx + s)$, $\phi_w(t)$ will be determined. $\phi_w(t)$ denotes the characteristic function of $w(x,Rx + s)$. $\phi_w(t)$ is absolutely integrable if

$$\int_{-\infty}^{\infty} \left| \phi_w(t) \right| dt < \infty .$$

$\phi_w(t)$ will be assumed to be absolutely integrable so that the following inversion theorem can be applied.

Inversion theorem: If the characteristic function $\phi_w(t)$ is absolutely integrable, then $w(x,Rx + s)$ obeys the continuous density function $f(w)$ where

$$f(w) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itw} \phi_w(t) \quad dt$$

and $w = w(x,Rx + s)$ which is defined by Equation 3-3.

The proof of this theorem will not be reproduced here; the details have been presented by Parzen (68, Ch. 9) among others.

A second theorem that will be applied several times in this section follows. While its proof is not complex, it is not commonly given in statistics textbooks and is given below.

Function theorem: The characteristic function of a linear combination of independent random variables is the product of the characteristic functions of the individual terms of the linear form.

Proof: Suppose that $L = \sum_{j=1}^{n} c_j v_j$ and that for j=1, ..., n the v_j's are independent random variables, the c_j's are constants, and $\phi_{v_j}(t)$ symbolizes the characteristic function of v_j.

$$\phi_{c_j v_j}(t) = \phi_{v_j}(c_j t) = E\left[\exp(itc_j v_j)\right]$$

$$\phi_L(t) = \phi_{\sum_{j=1}^{n} c_j v_j}(t) = E\left[\exp(itc_1 v_1 + \ldots + itc_n v_n)\right]$$

$$\phi_L(t) = E\left[\prod_{j=1}^{n} \exp(itc_j v_j)\right] = \prod_{j=1}^{n}\left[E(\exp(itc_j v_j))\right]$$

$$= \prod_{j=1}^{n}\left[\phi_{c_j v_j}(t)\right] = \prod_{j=1}^{n}\left[\phi_{v_j}(c_j t)\right]$$

Applying these two theorems one can determine $\phi_w(t)$. $\phi_w(t) = E\left[\exp(itw)\right] = E\left[\exp(it(b's + \frac{1}{2} x'Bs + k'x + \frac{1}{2} x'Kx))\right]$ where K and k' are both as defined just below Equation 3-3 in section B of this chapter. Now x will be replaced by x^o, the optimal level of the decision variables in Equations 3-1 and 3-2. From Equations 3-4 and 3-6, recall that

$$x^o = - K^{-1} k$$

and

$$w^o = w(x^o, Rx^o + s) = b's + \frac{1}{2} s' Bs - \frac{1}{2} k' K^{-1} k .$$

Then

$$\phi_{w^o}(t) = E\left[\exp(it(b's + \frac{1}{2} s' Bs - \frac{1}{2} k' K^{-1} k))\right]$$

The function theorem given above permits $\emptyset_{w^o}(t)$ to be expressed as follows:

$$\emptyset_{w^o}(t) = \emptyset_{b's}(t) \cdot \emptyset_{\frac{1}{2}s'Bs}(t) \cdot \emptyset_{-\frac{1}{2}k'K^{-1}k}(t)$$

Since all of the random variables are assumed to be independent or transformed to be so, the three factors of the right hand side of the formulation of $\emptyset_{w^o}(t)$ can be analyzed as shown below.

$$\emptyset_{b's}(t) = E\left[\exp(itb's)\right] = E\left[\exp(it \sum_{k=1}^{n} b_k s_k)\right]$$

$$= \prod_{k=1}^{n} \emptyset_{b_k s_k}(t) = \prod_{k=1}^{n} \int_{-\infty}^{\infty} \exp(itb_k s_k)\ f(b_k s_k)\ db_k s_k$$

$$\emptyset_{\frac{1}{2}s'Bs}(t) = E\left[\exp(its'Bs)\right] = E\left[\exp(it\tfrac{1}{2} \sum_{i=1}^{n} \sum_{k=1}^{n} s_i b_{ik} s_k\right] =$$

$$\prod_{i=1}^{n} \prod_{k=1}^{n} \emptyset_{s_i b_{ik} s_k}(t) = \prod_{i=1}^{n} \prod_{k=1}^{n} \int_{-\infty}^{\infty} \exp(it\tfrac{1}{2}s_i b_{ik} s_k)\ f(s_i b_{ik} s_k)\ ds_i b_{ik} s_k$$

$$\emptyset_{-\frac{1}{2}\ 'K^{-1}k}(t) = E\left[\exp(-it\tfrac{1}{2}k'K^{-1}K)\right] = E\left[\exp(-it\tfrac{1}{2} \sum_{j=1}^{m} \sum_{h=1}^{m} k_j K_{jh}^{-1} k_h)\right]$$

$$= \prod_{j=1}^{m} \prod_{h=1}^{m} \emptyset_{-\frac{1}{2}k_j K_{jh}^{-1}\ k_h}(t)$$

$$= \prod_{j=1}^{m} \prod_{h=1}^{m} \int_{-\infty}^{\infty} \exp(-\tfrac{1}{2}itk_j K_{jh}^{-1}\ k_h)\ \ f(k_j K_{jh}^{-1}\ k_h)\ \ dk_j K_{jh}^{-1}\ k_h$$

What is needed to derive these characteristic functions explicitly is to form the probability density function for each of the following products.

$$b_k s_k \quad k = 1, \ldots, n$$

$$s_i b_{ik} s_k \qquad i, k = 1, \ldots, n$$

$$k_j\ K_{jh}^{-1}\ k_h \qquad j, h = 1, \ldots, m$$

The density functions of these products are denoted above by $f(b_k s_k)$,

$f(s_i b_{ik} s_k)$, and $f(k_j\ K_{jh}^{-1}\ k_h)$, respectively. While it will obviously be tedious and unpleasant to do so, these probability density functions can be determined if the density functions for all of the b_i, s_i, b_{ik}, k_j, and K_{jh}^{-1} are known. The density functions of the products can be derived through the use of moment generating functions, convolutions, or by transforming the variables.

One of these methods can be used to determine the density functions for all i, k, j, and h of $b_k s_k$, $s_i b_{ik} s_k$, and $k_j K_{jh}^{-1}\ k_h$. Given these density functions, the characteristic functions, $\phi_{b_k s_k}(t)$ for all k, $\phi_{s_i b_{ik} s_k}(t)$ for all i and k, and $\phi_{k_j K_{jh}^{-1} k_h}(t)$ for all j and h, can be derived.

Since

$$\phi_{b's}(t) = \prod_{k=1}^{n} \phi_{b_k s_k}$$

$$\phi_{\frac{1}{2}s'Bs}(t) = \prod_{i=1}^{n} \prod_{k=1}^{n} \phi_{s_i b_{ik} s_k}(t)$$

$$\phi_{-\frac{1}{2}k'K^{-1}k}(t) = \prod_{j=1}^{m} \prod_{h=1}^{m} \phi_{-\frac{1}{2}k_j K_{jh}^{-1}\ k_h}(t)$$

one can determine $\phi_{b's}(t)$, $\phi_{\frac{1}{2}s'Bs}(t)$, $\phi_{-\frac{1}{2}k'K^{-1}k}(t)$, respectively. Multiplying these results together gives $\phi_{w^o}(t)$ to which the inversion theorem can be applied to determine $f(w^o)$. $f(w^o)$ is the representation to be used for $f(w)$. If this approximation is unsatisfactory, a numerical method can be used to find $f(w)$.

The method described in this section is very much dependent upon having or transforming the problem into one having independent random variables. Tintner, Sengupta, and their associates have studied the cases for linear programming problems extensively. These studies and their results depend heavily on the fact that an optimal solution to a linear programming problem will occur at a corner point on the feasible space. This condition does not hold for quadratic programming problems where interior optimal solutions are quite common. There is no method for solving a quadratic programming problem that is analogous to the complete description method for solving linear optimization problems.

Where the passive approach to stochastic programming is to be applied to a linear programming problem, the methods described by Tintner and Sengupta are directly applicable. To convert the problem delineated by Equations 3-1 and 3-2 into a linear programming problem define the matrices A, B, and C to be null matrices. Equation 3-1 becomes

$$w(x,y) = a' \; x + b' \; y$$

The constraint set is still

$$y = R\,x + s \quad \text{or} \quad -R\,x + y = s$$

where the vector y might be the slack variables that were added to a set of inequalities such as $- R\,x \leq s$. If y is a vector of slack variables, the elements of b will all be zeroes or very large numbers.

If the elements of the vectors a, b, and s and the elements of the R matrix are random variables, a joint density function

$$P\ (a,\ b,\ s,\ R)$$

is assumed. Then the density function of $w(x,\ R\,x + s) = a'\ b'\ R\,x + b'\ s$ is to be derived. According to Sengupta, Tintner, and Millham (75)

The approach assumes that in almost all possible situa-

> tions, i.e. for almost all possible variations of the parameters, the conditions of the simple nonstochastic linear program are fulfilled and the maximum achieved.

$w(x, R x + s)$ is to be optimized assuming no random variables as a first step. The optimal decision vector, $x^o = g(a, b, R, s)$ is then substituted into $w(x, R x + s)$ and the stochastic properties of the random variables are to be studied henceforth. Analogously it can be argued that the substitution of $x^o = - K^{-1} k$ for x in the derivation of $f(w)$ for the quadratic programming problem of this section is not completely unfounded.

D. The Active Approach to Stochastic Programming

The optimization problem to be considered in this section is still that of Equations 3-1 and 3-2 where some or all of the elements of a, b, s, A, B, C, and R are assumed to be random variables with known probability density functions. In the passive approach to stochastic programming, the probability density function of the preference function is derived. Although this is not the main purpose of the active approach, this can be achieved in this section too.

The active approach to stochastic programming has been studied by Tintner and Sengupta (75, 76, 78) among others. Before continuing, rewrite the problem of Equations 3-1 and 3-2 as follows.

$$\text{optimize:} \quad w(z) = d' z + \tfrac{1}{2} z' G z$$

$$\text{subject to:} \quad D z = s$$

$$\text{where} \quad d' = (a_1, \ldots, a_m, b_1, \ldots, b_n)$$

$$z' = (x_1, \ldots, x_m, y_1, \ldots, y_n)$$

$$s' = (s_1, \ldots, s_n)$$

$$G = \begin{pmatrix} A & C \\ C' & B \end{pmatrix} = (g_{ij}) \quad i,j = 1,\ldots,m,m+1,\ldots,m+n$$

$$g_{ij} = a_{ij} \quad i=1,\ldots,m \quad j=1,\ldots,m$$

$$g_{ij} = c_{ij} \quad i=1,\ldots,m \quad j=m+1,\ldots,m+n$$

$$g_{ij} = c_{ji} \quad i=m+1,\ldots,m+n \quad j=1,\ldots,m$$

$$g_{ij} = b_{ij} \quad i=m+1,\ldots,m+n \quad j=m+1,\ldots,m+n$$

$$D = (R \ -I_n) = (d_{ij}) \quad i=1,\ldots,n \quad j=1,\ldots,m+n$$

$$d_{ij} = r_{ij} \quad i=1,\ldots,n \quad j=1,\ldots,m$$

$$d_{ij} = -1 \quad i=1,\ldots,n \quad j = m + i$$

$$d_{ij} = 0 \quad i=1,\ldots,n \quad j = m + k \text{ for all } k \neq i$$

Note that $D\ z = I\ y - R\ x = s$ can be studied as the set of inequalities $-\ R\ x \leq s$ where the vector y is a set of slack variables, and I is the identity matrix.

The active approach to this stochastic programming problem would be to

optimize $\quad w(z) = d'\ z + \frac{1}{2}\ z'\ G\ z$

subject to $\quad D\ z^* = s^*\ U$

where d, z, G, and D are defined above. z^* is an m+n by m+n diagonal matrix whose elements on the diagonal are the elements of the vector z. Likewise, s^* is a diagonal matrix having the elements of s. Therefore,

$$z^* = z'\ I_{m+n}$$

$$s^* = s'\ I_n$$

where I represents the identity matrix. U is an n by m+n matrix with elements u_{ij} such that $u_{ij} \geq 0$ and $\sum_{j=1}^{m+n} u_{ij} = 1$.

Since $D z^*$ and $s^* U$ are equivalent n by m+n matrices,

$$d_{ij} z_j = s_i u_{ij}$$

The u_{ij}'s are controlled exogenously. u_{ij} is the percentage of the i th resource that will be allocated to the j th activity. The amount of that resource to be allocated to the j th activity is $s_i u_{ij}$.

Perhaps a more interesting formulation of the same problem is to consider the set of inequalities: $- R x \leq s$. Form the diagonal matrices of the x's and s's, $s^* = s' I_n$ and $x^* = x' I_m$. U is still the n by m+n matrix that was defined above. However, the significance of the u_{ij}'s is slightly different because the constraints are inequalities. Now u_{ij} is the minimum proportion of the i th resource that may be allocated to the non-slack activity x_j.

In the active approach to stochastic programming, the optimal solution will depend on the random variables (elements of a, b, A, B, C, R, and s) as well as the chosen u_{ij}'s. Since the u_{ij}'s are selected exogenously, it is possible to compare several alternative proportions or limits on the proportions of the s_i's. Suppose that a multitude of allocation matrices are chosen and labeled U^1, U^2, ..., U^k. If all possible U's are considered, the statistical distribution of w(x,y) can be analyzed. For problems where m and n are very small and computers are available, this may be a desirable method for deriving the probability density function of the objective function. Perhaps this should be defined as one of the numerical methods to be substituted for the passive approach in some cases.

E. Chance-Constrained Programming

Chance-constrained programming problems have been studied mainly by Charnes, Cooper, and their associates (8 through 13). In the notation of this chapter the problems can be characterized as follows:

$$\text{optimize} \quad w(x,y) \tag{Eq. 3-1}$$

$$\text{subject to} \quad \Pr\,(-R\,x + y = s) \geq \alpha \tag{Eq. 3-2a}$$

$$x = H\,s \tag{Eq. 3-2b}$$

Again some or all of the elements of a, b, A, B, C, R, and s are random variables. α represents the n by 1 vector, $(\alpha_1, \ldots, \alpha_n)'$, where α_i is the minimum acceptable probability that the constraint

$$-\sum_{j=1}^{n} r_{ij}x_j + y_i = s_i$$

holds. How close α_i is to zero should depend on the role and scarcity of the i th resource within the context of the decision problem. The matrix H is determined with reference to the constraint set, $-R\,x + y = s$. In some cases it may be possible to construct H from R via the methods for generalized inverses. R will not have an inverse in the usual sense since it is not a square matrix.

Most of the studies that have been completed on problems of this type are efforts to redefine the functions so that a deterministic equivalent problem can be solved. Charnes and Cooper (9) have shown three cases of particular interest. These have been called the E Model, the V Model, and the P Model. For the E Model, the problem is to maximize the expected value of w(x,y), $E[w(x,y)]$, subject to the constraints delineated by Equations 3-2a and 3-2b. The V Model involves the same constraints, but $E[w(x,y) - \bar{w}]^2$ is to be minimized. $\bar{w}$ is the target or desired value for

$w(x,y)$. The P Model requires that $Pr[w(x,y) > \bar{w}]$ be maximized; the constraint set remains as it was for the E and V Models. In each case the deterministic equivalent is a convex programming problem.

For each model the situation considered by Charnes and Cooper is a linear programming problem. In the notation of this chapter all of the elements of b, A, B, and C must be zeroes. The convex programming problems devised by Charnes and Cooper for optimizing

$$w(x) = a' x$$

subject to

$$Pr(- R x \leq s) \geq \alpha$$

and

$$x = H s$$

are conceived by using the following definitions.

$$\sigma_i^2(H) = E[r_i' H s - s_i]^2$$

$$u_i^2(H) = [E(s_i) - r_i' H E(s)]^2$$

r_i is the i th row of the matrix -R

$-K_{\alpha_i} = F^{-1}(\alpha_i)$ where F is the cumulative density function for a normal random variable with a zero mean and variance of one

The E Model is solved via the equivalent problem

to minimize $(Ea)' H (Es)$

subject to $u_i(H) - v_i \geq 0 \qquad i = 1, \ldots, m$

$$-K_{\alpha_i}^2 [\sigma_i^2(H)] + K_{\alpha_i}^2 [u_i^2(H)] + v_i^2 \geq 0 \qquad i = 1, \ldots, m$$

$$v_i \geq 0 \qquad i = 1, \ldots, m$$

The variables to be determined are the elements of H and the v_i $i=1,\ldots,m$. The elements of H are determined by the elements of R and s and will reflect the appearance of random variables in the original constraints, $-R\,x \leq s$.

The V Model is solved by determining the elements of H and all of the v_i's that

$$\text{minimize } E\left[a'\ H\ s - \overline{w}\right]^2$$

subject to the same constraints as those for the E Model.

Define $\overline{w}$ to be the desired value for $a'\ x$, i.e. $\overline{w} = \overline{a}'\ \overline{x}$. The convex programming formulation of the P Model is to

$$\text{maximize} \quad t\ v$$

subject to

$$\left[E(a)\right]\ t\ H\left[E(s)\right] - t\ v_o - t\left[E(\overline{a})\right] \geq 0$$

$$1 - E\left[a'\ t\ H\ s - t\ \overline{w}^{\ 2}\right] \geq 0$$

$$\overline{u}_i(h) - t\ v_i \geq 0 \qquad i = 1, \ldots, m$$

$$-K^2_{\alpha_i}\left[\overline{\sigma}^2_i(H)\right] + K^2_{\alpha_i}\left[\overline{u}^2_i(H)\right] + t^2\ v^2_i \geq 0 \qquad i = 1, \ldots, m$$

$$t\ v_i \geq 0 \qquad i = 1, \ldots, m$$

$$t \geq 0$$

where

$$\overline{\sigma}^2_i(H) = E\left[r'_i\ t \cdot H\ s - t\ s_i\right]^2$$

$$\overline{u}^2_i(H) = \left[t\ E(s_i) - r'_i\ t\ H\ E(s)\right]^2$$

This problem is also a convex programming problem; the variables are the same as those in the deterministic equivalents for the E and V Models.

In the E Model the expected value of the objective function is maximized; in the V Model the generalized mean square error is minimized. According to Charnes and Cooper (9), the motivation for the P Model is Simon's principle of satisficing (81, Chapters 14 and 15). Since $\Pr\left[w(s) \geq \overline{w}\right]$ is maximized, presumably one is satisfied by $w(x) = \overline{w}$ or $w(x) > \overline{w}$.

It should be noted that A. D. Roy (72) had previously described a problem that is similar to the P Model. Roy attempted to find a set of activity levels that would minimize the probability of the occurrence of a disaster. He assumed the first two moments of the joint density function of the variable activities to be known; the analysis proceeds by identifying activity combinations that will avoid a worse outcome than d--the disaster. Next Roy establishes the activity levels that provide the minimum upper bound on d.

In the 1960's numerous extensions and applications of chance-constrained programming have appeared. Hillier (43) has included zero-one variables in such a problem. Bertil Naslund has studied problems in capital budgeting under uncertainty (63). In another paper (62) Naslund has shown that under some conditions chance-constrained programming problems can be formulated as problems in variational calculus. Charnes, Cooper, and Thompson (12 and 13) have considered the relationship between chance-constrained programming and both critical path analyses and two-stage programming. Multi-stage programming, of which two-stage is the simplest case, will be discussed in the following section.

F. Multi-Stage Programming

The simplest example of a multi-stage programming problem is one that involves only two stages. The two-stage programming problem has been compared to both the chance-constrained problem (34, pp. 106-107) and the active approach to stochastic programming (34, p. 279). Therefore, this is the logical point at which to mention the two-stage problem. In this section the presentation will be carried out for the multi-stage situation of which the two-stage problem is a special case.

In a similar manner to Dantzig (18) the structure of the multi-stage programming problem can be illustrated as follows:

$$\text{optimize } w(x^1, x^2, \ldots, x^T \mid e^2, e^3, \ldots, e^T)$$

$$\text{subject to } s^1 = R_1^1 x^1$$

$$s^2 = R_1^2 x^1 + R_2^2 x^2$$

$$s^3 = R_1^3 x^1 + R_2^3 x^2 + R_3^3 x^3$$

$$\vdots$$

$$s^T = R_1^T x^1 + R_2^T x^2 + \ldots + R_T^T x^T$$

The superscripts identify within which of the m stages the vector or matrix is to be considered. s^1 is a vector of known constants; s^i, for all i between 2 and T, is a vector of random variables. In the i th stage the elements of s^i are functions of the random variable e^i, where e^i is an observation from a multivariate density function. The R_j^i 's are matrices of known constants for i,j = 1, ..., m. x^i is chosen to satisfy the set of constraints for the i th stage, given that all of the constraints for the previous stages are satisfied. The objective function is assumed to be convex; otherwise, most of the important theorems for optimization problems cannot be applied.

Dantzig describes (18) the solution process to be as follows. x^1 is chosen so that $s^1 = R_1^1 x^1$ is satisfied. x^2 must be such that $s^2 = R_1^2 x^1 + R_2^2 x^2$, given that $s^1 = R_1^1 x^1$ holds. x^3 must solve $s^3 = R_1^3 x^1 + R_2^3 x^2 + R_3^3 x^3$ while maintaining $s^1 = R_1^1 x^1$ and $s^2 = R_1^2 x^1 + R_2^2 x^2$. Finally x^T is chosen so that

$$s^T = \sum_{i=1}^{T} R_i^T x^i$$

assuming that the equalities for all of the previous stages are maintained. Dantzig shows that a T stage problem can be reduced to an T-1 st stage problem in a manner that is analogous to the methodology of dynamic programming. The T th stage is completed on the basis of the assumption of optimal levels for $x^1, \ldots, x^{T-1}$.

The two-stage problem is simply to

$$\text{optimize } w(x^1, x^2 \mid e^2)$$
$$\text{subject to } s^1 = R_1^1 x^1$$
$$s^2 = R_1^2 x^1 + R_2^2 x^2$$

The similarity of this problem to that of Equations 3-1 and 3-2 of this chapter is apparent if $x^1 = x$, $x^2 = y$, $R_1^2 = R$, $s^2 = s$, and $R_2^2 = I$ (where I is the appropriate sized identity matrix). $s^1 = R_1^1 x^1$ is simply an additional set of constraints on the decision variables. There is no reason why the objective function cannot be assumed to be linear, quadratic, or whatever is needed to represent the problem appropriately.

According to Dantzig, the existence of convex objective functions will permit the reduction of the T stage problem to an T-1 st stage problem, then to an T-2 nd stage problem, etc., and finally to a one stage problem. The difficulty is in finding these convex functions for each of these T reductions. The importance of their existence is that a local optimal solution will be a global optimal for the problem in which the convex functions are guaranteed to exist. The role of these functions will become more clear in the following section.

G. The Consideration of Time in Programming Problems

When some of the time dependent aspects of programming problems are considered, it is very possible to introduce issues that cannot be resolved. Two approaches will be delineated in this section. The first will be labeled programming over time to distinguish it from dynamic programming which is discussed later. Nemhauser (65) has emphasized that both of these alternatives are approaches to programming problems where time is explicitly considered and that neither provides an optimal solution to the problem under consideration.

Suppose that the value of the variable x is known for the previous period, the t-1 st, and it is desired to select optimal levels for x in periods t, t+1, t+2, ..., t+T. Assume that these levels must be selected at the beginning of period t; therefore, the optimal level of x at t+j, for any j between 0 and T, is determined with only the knowledge of x_{t-1}. Only when $j = 0$ is the value for x_{t+j} selected with the information concerning x_{t+j-1}. For programming over time the problem of the previous sections can be stated as follows.

$$\text{optimize } w(x_t, x_{t+1}, \ldots, x_{t+T}, y_t, y_{t+1}, \ldots, y_{t+T} \mid x_{t-1}, y_{t-1})$$

$$\text{subject to } \quad y_{t+j} = R_{t+j}\, x_{t+j} + s_{t+j} \qquad\qquad j = 0, \ldots, T$$

At time t the optimal levels of the variables x_{t+j} and y_{t+j} for $j=0,1,\ldots,T$ must be selected. It is this type of problem that Theil considers (89, Ch. 4) in extending his efforts on the problem delineated by Equations 3-1 and 3-2 when several time periods are to be considered. Theil includes a new set of variables and parameters for each time period. He redefines the matrices and vectors that were specified in section B of this chapter. Now

a and x are mT by 1 vectors

$$a = \left[a_j(t)\right] \quad , \quad x = \left[x_j(t)\right]$$

b, y, and s are nT by 1 vectors

$$b = \left[b_i(t)\right] \ , \ y = \left[y_i(t)\right] \quad , \ s = \left[s_i(t)\right]$$

R is an nT by mT matrix

$$R = \left[r_{ij}(t,\tau)\right]$$

A is an mT by mT matrix

$$A = \left[a_{jh}(t,\tau)\right] \ = \left[a_{hj}(t,\tau)\right]$$

B is an nT by nT matrix

$$B = \left[b_{ik}(t,\tau)\right] \ = \left[b_{ki}(t,\tau)\right]$$

C is an mT by nT matrix

$$C = \left[c_{ji}(t,\tau)\right]$$

$i, k = 1, \ldots, n \qquad j, h = 1, \ldots, m \qquad \text{and } t,\tau = 1, \ldots, T$

For this problem the w to be optimized is

$$w = \sum_{t=1}^{T} w(t) = \sum_{t=1}^{T}\left[\sum_{j=1}^{n} a_j(t)\ x_j(t) + \sum_{i=1}^{m} b_i(t)\ y_i(t)\right]$$

$$+ \tfrac{1}{2}\left[\sum_{t=1}^{T}\ \sum_{\Upsilon=1}^{T}\left[\sum_{j=1}^{m}\ \sum_{h=1}^{m} a_{jh}(t,\Upsilon)\ x_j(t)\ x_h(\Upsilon) + \sum_{i=1}^{n}\ \sum_{k=1}^{n} b_{ik}(t,\Upsilon)\ y_i(t)\ y_k(t)\right]\right]$$

$$+ \tfrac{1}{2}\left[\sum_{t=1}^{T}\ \sum_{=1}^{T}\left[\sum_{j=1}^{n}\ \sum_{k=1}^{n} c_{jk}(t,\Upsilon)\ x_j(t)\ y_k(\Upsilon) + \sum_{j=1}^{n}\ \sum_{k=1}^{n} c_{kj}(t,\Upsilon)\ x_j(t)\ y_k(\Upsilon)\right]\right]$$

subject to

$$y_i(t) = \sum_{P=1}^{t}\ \sum_{j=1}^{n} r_{ij}(t,P)\ x_j(P) + s_i(t)$$

for $i = 1, \ldots, n$ and $t = 1, \ldots, T$. For the linear programming problem $a_{jh} = b_{ik} = c_{jk} = 0$ for all of the i's, j's, k's, and h's.

Problems of this sort must be solved mathematically or intuitively whenever a budget or plan must be selected at time t for the next T periods. The purpose is to select x_{t+j} and y_{t+j} for all j, where $j = 0, 1, \ldots, T$. Unfortunately, the planner does not have the luxury of selecting each x_{t+j} and y_{t+j} with the knowledge of each respective x_{t+j-1} and y_{t+j-1}. That opportunity is afforded only for the first period, $j = 0$.

The second approach to be delineated in this section is that of dynamic programming. Studies on this subject have been popularized by Richard Bellman and his associates. Included within this dominion are many multi-stage problems of the sort that were described in the previous section of this chapter. In a dynamic programming problem the decision for period t+j is to be selected with the knowledge of the actual results or optimal allocations for period t+j-1 for all j, $j = 0, 1, \ldots, T$. In the framework of this chapter the problem is to

$$\text{optimize } w(x_{t+j}, y_{t+j} \mid x_{t+j-1}, y_{t+j-1})$$

$$\text{subject to } y_{t+j} = R_{t+j}\, x_{t+j} + s_{t+j}$$

for each of $j = 0, 1, \ldots, T$.

In the first approach described in this section there is one problem to be solved. This problem includes variables for all of the time periods under consideration. In the dynamic programming approach there are T+1 different problems to be solved--one for each of the periods in which a decision must be made. In each approach constraints other than a relation between y and x within the same time period (the t+j th) are likely to be important. Theil ignores relations such as

$$x_{t+j} = f_{t+j}\,(x_{t+j-1})$$

and

$$y_{t+j} = g_{t+j}\,(y_{t+j-1}, x_{t+j-1}) \; .$$

If one substitutes

$$\begin{aligned} x_{t+j-1} &= f_{t+j-1}\,(x_{t+j-2}) \\ x_{t+j-2} &= f_{t+j-2}\,(x_{t+j-3}) \\ &\;\;\vdots \\ x_t &= f_t\,(x_{t-1}) \end{aligned}$$

into

$$f_{t+j}\,(x_{t+j-1}) \, ,$$

x_{t+j} can be expressed as a composite function of only x_{t-1}. Likewise, y_{t+j} can be formulated to be dependent upon only x_{t-1} and y_{t-1}. The difficulty occurs when one tries to manipulate the composite functions F and G defined by

$$x_{t+j} = f_{t+j}\left[f_{t+j-1}\left\{f_{t+j-2}\left(f_{t+j-3} \cdots \left\{f_t(x_{t-1})\right\} \cdots\right)\right\}\right] = F\ (x_{t-1})$$

and

$$\begin{aligned} y_{t+j} &= g_{t+j}\left\{g_{t+j-1}\left[y_{t+j-2},\ f_{t+j-2}\ (\ x_{t+j-3}\ )\right]\right\} \\ &= g_{t+j}\left(g_{t+j-1}\left[g_{t+j-2}\left\{g_{t+j-3}\ (y_{t+j-4},\ x_{t+j-4})\right\},\ f_{t+j-2}(x_{t+j-3})\right] \cdots\right) \\ &\ \ \vdots \\ &= G(\ y_{t-1},\ x_{t-1}\ ). \end{aligned}$$

In this formulation y_{t+j} and x_{t+j} are formulated in terms of y_{t-1} and x_{t-1}, which are the initial conditions; this is known as the backward formulation (65, pp. 16-18). The forward formulation begins with x_{t-1} and y_{t-1}; the functions H and J are to be constructed so that

$$x_{t-1} = H\ (x_{t+j}) \text{ and } y_{t-1} = J\ (y_{t+j}\ ,\ x_{t+j}).$$

It should be obvious that neither the forward nor the backward formulation of the composite function will be simple. One may be inclined not to tackle this task for which he could not be blamed. However, when the number of computations needed to solve the dynamic programming formulation is compared to the number required for the apparently more direct methods, dynamic programming becomes more appealing. The savings via dynamic programming is considerable. Nemhauser has provided convincing evidence of this (65, p. 77). One of his examples is for a two-stage problem having ten state variables (y's) and ten decision variables (x's); direct programming methods require two thousand separate steps while the dynamic programming approach needs only two hundred eighty-nine. However, sometimes the direct approach of programming over time must be applied; these are problems in which decisions for period t+j must be determined prior to period t+j-1.

In this section only variables that are not random have been mentioned; it should be clear that when several time periods are to be considered, these problems are complex in themselves. After formulating such a problem, the methods that have been mentioned in this chapter may be applied. In the following two chapters some of these cases occur. In chapter five the parameters of the model described in chapter two will be estimated; in chapter six the solutions to simplified versions of the problems delineated in chapter three are provided.

V. THE ESTIMATED MODEL

A. Introduction

In this chapter the parameters that were specified in chapter two will be determined. This will show that, at least under certain assumptions, the model that has been delineated can be implemented. The numerical coefficients will then be substituted into the model so that the optimization problems can be considered; this will be completed in chapter six.

One might expect the methods of simultaneous equations to be applied to the multi-product model, but this will not be done. An extensive survey and discussion of the literature comparing single equation and simultaneous methods has been written by Alfred Field (30). Mr. Field has cited the studies by Christ (14, 15), Fox (31, 32, 34), Johnston (49), Quandt (71), and Waugh (94) among others. In most cases simple least squares is the single equation technique that is compared with simultaneous methods; the overwhelming conclusions are that the high cost of simultaneous estimation methods is rarely justified and that the single equation least squares estimates are not necessarily inferior when judged on statistical bases. For small samples very strong arguments have been given in favor of single equation least squares estimates.

Coefficients that are estimated by ordinary least squares can be shown to be best linear unbiased estimators (60). In matrix notation the general linear model is

$$y = z\,B + e \qquad \text{(Eq. 5-1)}$$

where there are n observations on the k independent variables (the z's) and the dependent variable, y. The estimated model is represented by

$$\hat{y} = z\,\hat{B} \qquad \text{(Eq. 5-2)}$$

and $e = y - \hat{y}$ represents the deviation between the true observed value of the dependent variable and the value that would be estimated to occur using $\hat{B}$ as the true value of B. The ordinary least squares estimators for B are the solutions for the B's that minimize e'e ($e'e = \sum_{i=1}^{n} e_i^2$). The usual assumptions are: (1) E(e) = 0 and $E(ee') = \sigma^2 I$; (2) the rank of z is k which must be less than n and implies that the k independent variables are independent among themselves; and (3) that the elements of z are non-stochastic. Since

$$e'e = y'y - 2\,B'\,z'\,y + B'\,z'\,z\,B \qquad \text{(Eq. 5-3)}$$

to minimize e'e is to minimize a quadratic function of the parameters of the regression equation (the elements of B). In this quadratic minimization problem there are no restrictions on the variables (the elements of B) and the B's that minimize e'e also maximize the sum of squares that is explained by the estimates.

For nearly all of the equations of the model (Equations 2-2 through 2-11) the assumptions of the previous paragraph will be presumed to hold. However, the production functions will be seen to have high multicollinearity, and ordinary least squares regression will not provide useful estimates of the parameters. Special assumptions will also be needed for the learning relationships.

Samples of twenty-one observations were obtained on nearly all of the major variables in the model. Because of the size of the samples and the strong arguments offered by the econometricians whom Mr. Field cites, only single equation methods will be applied here. After a brief description of the data that have become available, the estimates for each equation will be consummated.

B. The Data

A major American corporation has provided monthly data (January 1966-September 1967) for one multi-product plant on nearly all of the major variables that have been defined in this study. In return the author agreed not to publish any part of the actual data. In the plant various sizes and types of plastic products are produced. The estimated model will be a short run application since the data extend over such a brief horizon. It should be noted that products other than those considered here are produced in the plant; however, these products are incidental to the operation of the firm, and no inventory is maintained on these items.

Observations were collected on fifteen products, and these data were combined to be represented by ten outputs. This was necessary because some of the products replaced others in the production line. If it be apparent that one replaced another, these two were combined so that enough observations would be available for single equation parameter estimates. When one considers the allocation of the factors among the various outputs, such combinations of outputs is not unreasonable. The combined products would have played similar roles in relation to the other products during the various months.

The types of machinery and labor each were aggregated into single conglomerates so that the degrees of freedom for the explained sum of squares is small when compared to the degrees of freedom associated with the sum of squares due to error. There is only one material used in the production process. In the notation delineated in chapter two,

m - the number of materials is one,

k - the number of types of labor being considered is one, and

h - the number of types of machinery used is one.

Data (not always non-zero) were available for each product (one through ten) for each month (one to twenty-one) in thousands of units for each of the following variables: the level of output, q; the quantity of the output shipped, q^s; the level of finished goods inventory for each output, q^*; and the maximum and minimum levels that q^* should take according to the contracts with customers. The minimum of this range will be used as an estimate for q^d (the target level for q^*). The prices for each product for each month were obtained from the data on the sales value and quantities shipped for each individual item.

Although data were not specified for the labor hours that were applied to each product separately, these observations on the individual elements of x^L could be determined. From labor information the ratios of production to labor hours allocated were obtained for each product; these represent estimates for the average product of labor with respect to each output. These estimates are given in Table 1. Multiplying the average product of labor and the output level gives the labor hours applied to each finished good in thousands of hours.

For the material that is used in the plastics production, aggregate observations were available for each month on each of the following: the quantity used, $\overline{X}$; the quantity received, X; and the beginning inventory, X^*. These aggregated figures are sufficient to estimate the parameters of Equations 2-7 and 2-8 of the model. However, the quantity of materials used in production must be allocated among the products to estimate the production functions for each of the ten products.

Given the total number of pounds of the material that was used during each month, this total was distributed among the various products that were produced in any particular period. This allocation was made on the basis of the weight of the product being considered and the number of units that were produced. Since there is one material, m = 1, $\bar{x}$ will be a column vector of ten elements (one for each product). The data that were given are $\sum_{i=1}^{10} \bar{x}_i$ and the weights of the products (w_i, i = 1, ..., 10).

Letting q_i be the level of output of the i th product for the period being considered, the individual $\bar{x}_i$'s are determined by

$$\bar{x}_i = \left(\frac{w_i \; q_i}{\sum_{i=1}^{10} w_i q_i} \right) \sum_{i=1}^{10} \bar{x}_i$$

for each product (i=1,...,10) and each period. Note that if no units of the i th product are produced in a period $q_i = 0$ and $\bar{x}_i = 0$ even though $\sum_{i=1}^{10} \bar{x}_i$ and $\sum_{i=1}^{10} w_i q_i$ are not zero for that month.

The number of machine hours that were applied to each product in each month were directly provided. These were the twenty-one observations on each of the elements of the vector x^M.

The average products for labor, machine hours, and materials were estimated for those given in Table 1. Note that the average product of labor is larger than the other two average product estimates for all ten products. This will be significant in the following section where the estimates are determined for the marginal products of these inputs.

Table 1. Average products

Product number	Average product of labor	Average product of machine hrs.	Average product of materials
1	9.075	0.484	3.911
2	8.333	0.508	0.512
3	12.666	0.680	2.953
4	7.847	0,807	6.337
5	12.857	0.781	1.264
6	8.945	0.386	0.574
7	8.353	0.747	4.430
8	8.869	0.531	2.364
9	17.001	0.445	1.430
10	10.638	0.643	5.025

The prices of labor and materials, (v and r, respectively) as well as the assessment for each machine hour (y) were considered by the firm to be fixed for the short horizon over which the data were collected. The micro-economic theorist may be inclined to conclude that the markets in which these inputs were purchased are perfectly competitive. However, this is not necessarily the case since a fixed price is a conclusion derived from the assumptions of perfect competition; a fixed price could be called a necessary occurrence in a perfectly competitive market. This does not imply that all inputs purchased at a fixed price by one firm are being obtained in a market where new sellers enter and leave freely or where the seller obtains a fixed price for all that he has available. More likely the limited data that are available for the twenty-one months represent one horizontal portion of the firm's input demand curves; these portions are most likely non-increasing step functions.

For a period as short as twenty-one months it is likely that there are contract agreements on wage rates and the price of a pound of the material.

For the accounting life of a machine, the cost assessment is normally fixed upon the purchase of the machine; usually tax laws prohibit a firm from varying the depreciation method during the accounting life of a fixed asset.

C. Estimating the Production Functions

1. Demonstrating the multicollinearity

Efforts in estimating the parameters of a production function are now nearly ancient. Professor Walters' study (93) which was cited in chapter two, mentions most of the well known papers on this subject. Since capital, labor, and output will be included within the structure of most meaningful production functions, the trends of the capital-output and capital-labor ratios are engrained in the estimation of a production function. This is not an idle comment because a constant capital-labor ratio for a firm, industry, or nation will imply that two independent variables in the production function are not independent of each other. This is a transgression upon the second assumption of single equation least squares regression as stated in the introduction to this chapter. For each set of two independent variables (columns of z in Equation 5-1) that are not independent of each other, the rank of z is reduced by one. If there are p sets of columns of z that are related, the rank of z is at most k-p. If the rank of z is less than k, there is multicollinearity among the independent variables.

Upon beginning to apply single equation least squares to the data on q, x^L, x^M, and $\overline{x}$, the correlation matrices were found to be those given in Table 2. The form of the correlation matrix is:

	labor	machine hrs.	materials
labor	1.000	r_{12}	r_{13}
machine hrs.	r_{12}	1.000	r_{23}
materials	r_{13}	r_{23}	1.000

From looking at these correlation matrices the question of whether or not multicollinearity is important in the estimation of the production functions for this study should be resolved. Multicollinearity is very much a problem in these data on labor, machinery, and materials. Some of this correlation is a result of the ways in which the data were obtained, but not all of the blame lies here. People from the corporation agreed that these high correlations among the inputs were logical and appropriate. The effects of highly correlated independent variables on single equation least squares estimates have been discussed by Fox (33) and Goldberger (38) among others. Each has emphasized two points: (1) the regression estimates are unreliable; and (2) the t statistics for the estimates will often be irrelevant.

The question to be considered at this point is how to alleviate the situation. Within this context it should be realized that estimating a production function for nearly any firm will involve the problem of multicollinearity. This must be expected because often the data for one independent variable will be obtained using the observations available for another. Even if multicollinearity is not introduced in this fashion, nearly constant capital-labor and capital-materials ratios can be expected for a firm. It is not only the particular set of data used in this application that causes single equation ordinary least squares estimates to

Table 2. Correlation matrices for production functions

Product #1	R^2 = .998		Product #2	R^2 = .999	
1.000	.991	.892	1.000	.989	.874
	1.000	.923		1.000	.835
		1.000			1.000
Product #3	R^2 = 993		Product #4	R^2 = .997	
1.000	.264	.938	1.000	.992	.706
	1.000	.441		1.000	.718
		1.000			1.000
Product #5	R^2 = .998		Product #6	R^2 = .999	
1.000	.650	.555	1.000	.835	.497
	1.000	.916		1.000	.343
		1.000			1.000
Product #7	R^2 = .998		Product #8	R^2 = .996	
1.000	.982	.822	1.000	.992	.816
	1.000	.767		1.000	.829
		1.000			1.000
Product #9	R^2 = .994		Product #10	R^2 = .995	
1.000	.992	.921	1.000	.847	.908
	1.000	.929		1.000	.791
		1.000			1.000

provide useless estimates for the marginal products. One should expect to need another estimation technique to determine any firm's production coefficients.

To estimate functions whose independent variables are highly correlated, the most common method is to change the list of independent variables. This can be done by simply dropping one of each pair of the correlated variables. An alternative procedure is to use stepwise regression. In either instance, however, not all of the original independent variables will appear in the estimated equations. Most methods for handling multicollinearity have this limitation. This may not be a serious constraint if the purpose of the study is to explain the observed values of the dependent variables. However, the purpose for estimating the production functions in this effort is to relate all of the inputs that are utilized to the output levels. Here a model involving certain predetermined, relevant variables is being constructed. Recall that the production function with all of the inputs that are used in the process is to become a constraint in the optimization problems to be studied in the following chapter. Therefore, suggestions that allow variables to be eliminated while handling multicollinearity are not very useful to this study.

Rather than changing the variables in the estimation problem, more information on the specific function to be estimated will be considered. When single equation estimation techniques are presumed, the slope coefficients of the production functions are the marginal productivity estimates for the respective inputs. Therefore, additional information is evident on these parameters; they should be positive. Upon examining the regression coefficients for the simple least squares estimates of the marginal productivities, several problems appear. Eight of the ten outputs show one or more negative slope coefficient. Also for each of the ten products the marginal product of labor estimated by simple least squares is statistically the most significant coefficient. For seven of the ten products the marginal product of labor is the only one that is statistically significantly positive. While negative marginal products are not impossible or uninteresting, this many non-positive marginal products for materials is a telling tale.

Goldberger (38), Zellner (95, 96), Boot (5), and Fox, Sengupta, and Thorbecke (34) all have suggested that a priori information be collected on parameters wherever multicollinearity appears to be a problem. Such a situation does exist when one tries to estimate linear production functions for a single plant. It should be possible to identify whether the average products (AP) of the inputs are above or below the respective marginal products (MP). From basic micro-economic theory, if MP is larger than AP, the use of the particular input is defined to be in stage one. If all of the AP's are above their respective MP's, the intercept of the production function (coefficient a in Equation 2-1) will be negative; this implies that a constant percentage increase of the use of each input will give a larger percentage increase in the output. This is unlikely for any established production process and can be verified for a particular case by the plant manager. Thus, constraints for the parameters of the production function can be determined a priori of the specific observations on the inputs and outputs.

For the firm to which the model of chapter two is being applied, it was determined that the relevant stage of utilization of all three inputs was stage two, and therefore, $0 < MP < AP$ for all inputs. A positive intercept ($a > 0$) together with these three constraints imply that the law of diminishing returns has become active for the utilization of each input.

2. <u>The estimation procedure</u>

Although the case against single equation simple least squares is quite strong, this does not preclude one from seeking a set of marginal products and an intercept for the production function that will explain most of the total sums of squares of the output level using all of the inputs that are relevant. The technique to be considered here is very close to that of restricted least squares, if it is not a special, limited case of this technique. Restricted least squares estimates would be a set of parameters that would minimize $e'e$ of Equation 5-2 subject to the a priori information on the elements of B, where the vector B is a vector of marginal products.

The single equation restricted least squares problem for each product is to minimize

$$\sum_{i=1}^{N} e_i^2 = \sum_{i=1}^{N} y_i^2 - 2 \sum_{j=0}^{3} B_j \left(\sum_{i=1}^{N} a_{ij} y_i \right) + \sum_{j=0}^{3} \sum_{h=0}^{3} B_j a_{jh} a_{hj} B_h \qquad \text{(Eq. 5-4)}$$

subject to $$0 < B_o \qquad \text{(Eq. 5-5)}$$

$$0 < B_j < H_j \qquad j=1,2,3 \qquad \text{(Eq. 5-6)}$$

where H_j is the average product of the j th input, B_j is the marginal product of the j th input, and B_o is the intercept of the production function. Many persons have suggested this method for utilizing a priori information in single equation regression problems. Among them are several of the references cited earlier in this chapter (5, 34, 38, 95).

The optimal solution to a problem such as that delineated by Equations 5-4, 5-5, and 5-6 will often involve setting the variables at a boundary point. Because of this, to consider only the programming problem of Equations 5-4, 5-5, and 5-6 is not satisfactory to estimate the marginal products and intercepts for the production process. Too often the intercept would be zero, implying constant returns to scale, and the marginal products would be zero meaning that the last units of the input were employed unproductively. Two adjustments were made.

For the intercept a lower bound of 1.000 was set, $1.000 < B_o$, because the minimum non-zero observation on the quantity produced was 1.0. To consider the ranges of values for the marginal products (B_j's, j=1,2,3) between the boundary points, several lower bounds were selected. For each product six sets of constraints were considered. They were as follow:

(i) $B_j \geq H_j$ j=1,2,3 (Eq. 5-7)

(ii) $B_j \geq .5H_j$ j=1,2,3

(iii) $B_j \geq .25H_j$ j=1,2,3

(iv) $B_j \geq .125H_j$ j=1,2,3

(v) $B_1 \geq .100$, $B_2 \geq .010$, $B_3 \geq .050$

(vi) $B_j \geq .0001$ j=1,2,3

$B_o \geq 1.000$ for all six cases

H_j is the average product for the j th input. j=1 for labor, j=2 for machine hours, and j=3 for materials. Shortly it will be seen that estimates for case (i) never occur; this validates the information from the firm that $MP_j < AP_j$ for j=1,2,3.

Another set of constraints were selected to ensure the statistical significance of the parameter estimates. It is true that this may be too restrictive for studies in which the purpose is to explain the sum of squares due to the dependent variable. However, since the parameter estimates of this chapter are to become coefficients in the programming models for the firm, a bit of conservatism does not seem to be unwarranted. A set of constraints have been included to guarantee that all of the marginal products and intercepts will be significantly larger than zero with ninety-five per cent confidence. In other words, the ratio of the estimate to its standard error must be greater than the relevant one tail t-statistic at the ninety-five per cent level. If this is the case, the null hypothesis of $B_j > 0$ cannot be rejected for j=0,1,2, and 3. Therefore,

$$\frac{B_j}{s_{B_j}} \geq t_{\alpha} \qquad \text{(Eq. 5-8)}$$

Within the confines of Equation 5-8 and the relevant case of Equation 5-7, the objective function given by Equation 5-3 was minimized. The minimum sum of squares error (SSE) when restricted by Equations 5-7 and 5-8 was larger than the unrestricted minimum (obtained from simple least squares estimates) for each of the ten products. However, the relevant question is to what extent was a loss incurred. That the increase in the unexplained sums of squares (SSE) is hardly significant will be apparent when the R-squares are compared for the restricted and unrestricted minima. Considering the gain in economic content and statistical significance of the parameters, the increase in SSE can be ignored.

A result that is similar to the conclusion of the previous paragraph was proved by Professor Sengupta for a study on stochastic programming where the variance of a random variable is to be minimized (77). In that study the additional information on the expected value of the random variable was used in a way similar to the use of the t statistic above.

Some of the properties of the restricted least squares estimators can be shown without too much difficulty. The expected values and the variances of the estimators are derived below since these are needed to calculate the t-statistics for each B_j and to see if the estimators are unbiased. The problem delineated by Equations 5-4, 5-5, 5-6, and 5-7 can be formulated with Lagrangian multipliers. Feasible solutions will be those that provide positive slack variables, positive B_j's, and involve either the j th slack variable or the j th Lagrangian multiplier (the one multiplying the j th constraint).

Before continuing, it should be emphasized that the method to be described is very much dependent upon the constraints determined by Equations 5-5 and 5-6. In particular it is the restriction that none of the four B_j's may be zero that allows this special case of the Kuhn-Tucker Theorem to be applied. Otherwise the first set of restrictions given below in Equation 5-9 are inequalities. (Hadley has labeled this special case the set $j \varepsilon J$, (40, pp. 191-192).)

The Lagrangian function can be formed as

$$L = y'y - 2\,B'\,z'\,y + B'\,z'\,z\,B + \lambda(B - H + W)$$

where λ is a row vector of Lagranian multipliers, H is a column vector of upper bounds for B, and W is a column vector of positive slack vectors with elements W_i; $W_i = v_i^2 \geq 0$. The feasible solutions must solve the linear equations

$$\begin{bmatrix} G & I & \emptyset \\ I & \emptyset & -I \end{bmatrix} \begin{bmatrix} B \\ W \\ \lambda' \end{bmatrix} = \begin{bmatrix} z'\,y \\ H \end{bmatrix} \qquad \text{(Eq. 5-9)}$$

where

I is the 4 by 4 identity matrix,

$\emptyset$ is a 4 by 4 null matrix, and

G is the matrix z'z.

In addition to the constraints of Equation 5-9, the following equalities must be maintained.

$$\lambda_i W_i = \lambda_i v_i^2 = 0 \text{ for all } i \qquad \text{(Eq. 5-10)}$$

From the last set of constraints, Equation 5-10, it is obvious that either the i th slack variable or the i th Lagrangian multiplier is zero for i=1,2,3,4. Each time that this consideration is applied to the set of equations represented by Equation 5-9 a set of eight equations in eight unknowns will remain. Four of the unknowns will be the B_j's.

Denote the four variables other than the B's by the vector Z so that the eight equations in eight unknowns can be represented by

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} B \\ Z \end{bmatrix} = \begin{bmatrix} z'y \\ H \end{bmatrix}$$

where

$A_{11} = z'z$

$A_{21} = I_4$

A_{12} is a 4 by 4 constructed from $\begin{bmatrix} I_4 & \emptyset \end{bmatrix}$ by deleting either the i+4 th or i+8 th column for i=1,2,3,4, and

A_{22} is a 4 by 4 constructed from $\begin{bmatrix} \emptyset & -I_4 \end{bmatrix}$ by deleting either the i+4 th or i+8 th column for i=1,2,3,4. Then

$$\begin{bmatrix} B \\ Z \end{bmatrix} = \begin{bmatrix} (z'z)^{-1} & A_{12} \\ I_4 & A_{22} \end{bmatrix} \begin{bmatrix} z'y \\ H \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} \begin{bmatrix} z'y \\ H \end{bmatrix}$$

where

$$C_{11} = (z'z)^{-1} (I_4 + A_{12}D^{-1} I_4(z'z)^{-1})$$

$$C_{12} = -(z'z)^{-1} A_{12} D^{-1}$$

$$C_{21} = - D^{-1} A_{21} I_4$$

$$C_{22} = D^{-1}$$

and

$$D = A_{22} - A_{12} (z'z)^{-1} I_4$$

This can be shown by applying the inversion method for partitioned matrices that is reviewed by Goldberger (38, p. 27). Therefore,

$$B = C_{11} z'y + C_{12} H$$

$$B = (z'z)^{-1} (z'y) + (z'z)^{-1} A_{12} D^{-1} I_4 (z'z)^{-1} (z'y) + C_{12} H$$

Substituting $y = z B + e$, replacing $(z'z)^{-1} (z'z)$ by I_4 and removing I_4 where it is not needed leaves

$$B = B^T + (z'z)^{-1} z' e + (z'z)^{-1} A_{12} D^{-1} B^T +$$

$$(z'z)^{-1} A_{12} D^{-1} (z'z)^{-1} e + C_{12} H$$

where B^T is the true value of B, such the $y = z B^T$. Since $E(e) = 0$,

$$E(B) = B^T + (z'z)^{-1} A_{12} D^{-1} B^T + C_{12} H.$$

The estimate of B will be biased unless $(z'z)^{-1} A_{12} D^{-1} = 0$ and $C_{12} H = 0$.

The variance-covariance matrix of B is labeled COV(B) below.

$$COV(B) = E(\ W\ ee'\ B'\ ) = W\ W'\ ^{2}$$

assuming that $E(e'e) = \sigma^2\ I_4$ is still valid and letting W be defined as:

$$W = (z'z)^{-1}\left[I + A_{12}D^{-1}\ (z'z)^{-1}\right]\ z' .$$

The computation of the t-statistics to test the null hypothesis $B_j > 0$ is

$$t_j = \frac{B_j}{\sqrt{w_{jj}(MSE)}} \qquad j=1,2,3,4.$$

where w_{jj} is the j,j th element of W W' and MSE is the mean squared error for the multiple regression (SSE/N-4), and B_j is the estimated value for the marginal product.

3. The estimates

The results of the estimation procedure follow. For each case of a particular product the parameters, with the t statistics in parentheses, that minimize the sum of squares due to error are given; these parameters must be within the two sets of constraints--Equations 5-7 and 5-8. For the programming model one case must be selected for each product. The selected set is identified by underlining the case number. In other words, a production subphase within phase two is being recommended. The single case is chosen from among the six for each product by picking that case which has the minimum squared error. The assumption that the relevant phase of production is phase two seems to be verified since there are no feasible parameters for case (i) for any product such that SSE > 0.

One additional case was investigated for each product; it is identified in Table 3 is case (vi). This case was very close to constraining the mar-

Table 3. The production functions

case	intercept	mp-labor	mp-machine hrs.	mp-materials	R^2	F	SSE
			Product 1 total degrees of freedom = 12 (no estimates for cases i or ii)				
iii	1.000	2.269	.121	1.002	.981	103.3	3780.4
	(3802.8)	(3838.2)	(3495.7)	(9.0)			
iv	1.000	1.134	.061	1.596	.959	46.9	8137.6
	(2591.9)	(1308.0)	(1192.3)	(9.8)			
v	1.000	.100	.010	2.114	.933	27.8	13372.2
			Product 2 total degrees of freedom = 13 (no estimates for case i)				
ii	1.000	4.167	.254	.256	.999	2085.4	213.9
	(99964.6)	(275192.1)	(11.5)	(2.5)			
iii	1.000	2.083	.400	.408	.997	836.2	532.6
	(63353.0)	(87198.6)	(11.5)	(2.5)			
iv	1.000	1.042	.473	.484	.996	596.2	746.1
	(53523.8)	(36834.7)	(11.5)	(2.5)			
v	1.000	.010	.539	.553	.995	457.9	970.4
	(46932.7)	(3100.2)	(11.5)	(2.5)			

Table 3 (Continued)

case	intercept	mp-labor	mp-machine hrs.	mp-materials	R^2	F	SSE
Product 3 total degrees of freedom = 18 (no estimates for cases i, ii)							
<u>iii</u>	1.000 (5732.0)	3.167 (34452.1)	.170 (1.75)	.738 (34804.8)	.827	16.8	98230.9
iv	1.000 (3970.5)	1.583 (11943.2)	.354 (2.5)	.369 (12066.9)	.641	6.2	204357.5
v	1.000	.100	.517	.050	.410	2.4	335467.6
Product 4 total degrees of freedom = 14 (no estimates for cases i, ii)							
<u>iii</u>	31.680 (2.2)	1.962 (658.9)	.202 (725.3)	.737 (3.0)	.975	99.0	9982.8
iv	50.260 (2.3)	.981 (213.2)	.101 (234.7)	1.167 (3.1)	.941	40.0	23836.5
v	66.974 (2.3)	.100 (16.5)	.010 (17.6)	1.554 (3.2)	.898	21.9	41500.2

Table 3 (Continued)

case	intercept	mp-labor	mp-machine hrs.	mp-materials	R^2	F	SSE
			Product 5 total degrees of freedom = 14 (no estimates for cases i, ii)				
iii	15.431	3.214	.195	.316	.992	313.2	2508.6
	(2.4)	(2489435.0)	(134022.0)	(3.9)			
iv	1.000	5.921	.098	.461	.993	326.8	2405.0
	(85642.3)	(10.8)	(1497507.0)	(4.9)			
v	1.000	6.549	.010	.725	.984	149.1	5224.5
	(58106.5)	(8.1)	(103994.8)	(5.2)			
			Product 6 total degrees of freedom = 18 (no estimates for case i)				
ii	1.000	4.473	.193	.287	.993	466.6	8337.0
	(18168.2)	(3027893.0)	(6.7)	(2.1)			
iii	1.000	2.236	.291	.433	.983	204.5	18842.0
	(12085.4)	(1007052.9)	(6.7)	(2.2)			
iv	1.000	1.118	.390	.506	.977	149.1	25679.0
	(10351.9)	(431337.3)	(6.7)	(2.1)			
v	1.000	.100	.384	.573	.971	115.9	32826.0
	(9155.6)	(34119.5)	(6.7)	(2.1)			

Table 3 (Continued)

case	intercept	mp-labor	mp-machine hrs.	mp-materials	R^2	F	SSE
			Product 7 total degrees of freedom = 18 (no estimates for cases i, ii, iii)				
iv	1.000	1.044	.094	1.726	.953	71.0	47402.1
	(16520.6)	(65730.2)	(168665.2)	(11.9)			
v	1.000	.100	.599	.050	.990	330.5	10567.2
	(120829.9)	(6305.9)	(35.1)	(100226.4)			
vi	1.000	.0001	.618	.0001	.989	302.0	11551.8
	(115580.9)	(6.0)	(34.6)	(192.3)			
			Product 8 total degrees of freedom = 20 (no estimates for cases i, ii)				
iii	31.830	2.217	.133	.591	.986	291.8	10863.1
	(3.0)	(289010.9)	(162214.8)	(5.3)			
iv	50.630	1.109	.066	.964	.967	117.0	26544.8
	(3.1)	(92446.4)	(51886.6)	(5.6)			
v	67.124	.100	.010	1.290	.942	65.5	46236.5
	(3.1)	(6319.6)	(5921.7)	(5.6)			

Table 3 (Continued)

case	intercept	mp-labor	mp-machine hrs.	mp-materials	R^2	F	SSE
			Product 9 total degrees of freedom = 19 (no estimates for cases i, and ii)				
iii	1.000 (1167.9)	4.251 (756.9)	.111 (692.2)	.358 (3997.0)	.982	204.4	3104.1
iv	38.256 (4.4)	2.126 (1057.3)	.056 (1404.6)	.179 (7077.4)	.870	25.1	22421.1
v	66.138 (4.8)	.101 (31.4)	.010 (157.4)	.050 (121.1)	.664	7.4	57938.8
			Product 10 total degrees of freedom = 11 (no estimates for cases i, ii, and iii)				
iv	43.443 (3.2)	1.330 (26293.0)	.080 (183363.9)	.628 (30979.1)	.908	17.2	15186.3
v	1.000 (60003.9)	8.332 (19.7)	.010 (81638.8)	.050 (57301.1)	.984	106.1	2669.1
vi	1.000 (139915.7)	5.375 (21.1)	.194 (13.7)	.0001 (350.6)	.999	2611.3	110.2

ginal products to be only positive. For case (vi), $B_j \geq .0001$ j=1,2,3, and $B_0 \geq 1.000$. Case (vi) was found to be inferior to case (v) for all products except for number ten. For product ten the sixth case was the one that minimized SSE; however, only one marginal product was at the lower bound.

Finally the production functions to be entered into the programming model are specified. These will be the ten production functions formulated from the underlined cases for each product in Table 3. In the framework of Equation 2-1 the ten intercepts form a ten element column vector a; the ten marginal products form the diagonal matrix A^L; the ten marginal products of machine hours are the elements of the diagonal matrix A^M; and the diagonal matrix $\bar{A}$ is formed by the ten marginal products of materials.

$$a = \begin{bmatrix} 1.000 \\ 1.000 \\ 1.000 \\ 31.680 \\ 15.431 \\ 1.000 \\ 1.000 \\ 31.830 \\ 1.000 \\ 1.000 \end{bmatrix}$$

$$A^L = \begin{bmatrix}
2.269 & 0.0 & & & \cdots & & & & & 0.0 \\
0.0 & 4.167 & 0.0 & & \cdots & & & & & 0.0 \\
0.0 & 0.0 & 3.167 & 0.0 & & \cdots & & & & 0.0 \\
0.0 & 0.0 & 0.0 & 1.962 & 0.0 & \cdots & & & & 0.0 \\
0.0 & \cdots & & 0.0 & 5.921 & 0.0 & \cdots & & & 0.0 \\
0.0 & \cdots & & & 0.0 & 4.473 & 0.0 & \cdots & & 0.0 \\
0.0 & \cdots & & & & 0.0 & 0.100 & 0.0 & 0.0 & 0.0 \\
0.0 & \cdots & & & & & 0.0 & 0.100 & 0.0 & 0.0 \\
0.0 & \cdots & & & & & & 0.0 & 4.251 & 0.0 \\
0.0 & \cdots & & & & & & & 0.0 & 5.375
\end{bmatrix}$$

$$A^M = \begin{bmatrix} .121 & 0.0 & & & \cdots & & & & & 0.0 \\ 0.0 & .254 & 0.0 & & \cdots & & & & & 0.0 \\ 0.0 & 0.0 & .170 & 0.0 & & \cdots & & & & 0.0 \\ 0.0 & 0.0 & 0.0 & .202 & 0.0 & & \cdots & & & 0.0 \\ 0.0 & \cdots & & 0.0 & .098 & 0.0 & & \cdots & & 0.0 \\ 0.0 & \cdots & & & 0.0 & .193 & 0.0 & & \cdots & 0.0 \\ 0.0 & \cdots & & & & 0.0 & .599 & 0.0 & 0.0 & 0.0 \\ 0.0 & \cdots & & & & & 0.0 & .010 & 0.0 & 0.0 \\ 0.0 & \cdots & & & & & & 0.0 & .111 & 0.0 \\ 0.0 & \cdots & & & & & & & 0.0 & .194 \end{bmatrix}$$

$$\bar{A} = \begin{bmatrix} 1.002 & 0.0 & & & \cdots & & & & & 0.0 \\ 0.0 & .256 & 0.0 & & \cdots & & & & & 0.0 \\ 0.0 & 0.0 & .738 & 0.0 & \cdots & & & & & 0.0 \\ 0.0 & 0.0 & 0.0 & .737 & 0.0 & \cdots & & & & 0.0 \\ 0.0 & \cdots & & 0.0 & .461 & 0.0 & & \cdots & & 0.0 \\ 0.0 & \cdots & & & 0.0 & .287 & 0.0 & & \cdots & 0.0 \\ 0.0 & \cdots & & & & 0.0 & .050 & 0.0 & 0.0 & 0.0 \\ 0.0 & \cdots & & & & & 0.0 & 1.290 & 0.0 & 0.0 \\ 0.0 & \cdots & & & & & & 0.0 & .358 & 0.0 \\ 0.0 & \cdots & & & & & & & 0.0 & .0001 \end{bmatrix}$$

D. Estimating the Input Demand Curves

It has already been argued that fixed input prices may exist for inputs that are not purchased in markets that are perfectly competitive. Recall that both machinery usage and the labor input have been aggregated (k = 1 and h = 1). This will mean that the allocations of input costs to the various products will not be straightforward. For instance, if a product requires the use of two machines whose assessed costs per hour are different, the cost will be presumed to be that of the higher costing machine.

From the records on labor standard information, labor costs per hour are obtainable. The labor cost data is updated each year on May 1st; however, the changes from 1965 to 1967 were less than five per cent per hour. For this reason the costs that became effective in May of 1967 will be assumed. This suggests that the labor assessments will not be underestimated. Furthermore, these are the most useful assessments for prediction of costs for future months.

The costs below are per labor hour as applied to products one through ten.

$$v = \begin{bmatrix} 16.160 \\ 16.160 \\ 18.180 \\ 16.160 \\ 16.160 \\ 16.160 \\ 16.160 \\ 16.160 \\ 16.160 \\ 16.160 \end{bmatrix} = g^L$$

The slope matrix of the input cost is taken to be a null matrix. This must be if labor costs are assessed using standard information.

The machine costs to be allocated to this input for each output are based on the per hour operating costs that were available for the machines. The operating costs include depreciation, taxes, space, systems, insurance, and maintenance assessments. Many of the outputs require the use of only one type of machine. Products #1, #2, #4, #6, #7, #8, and #10 use only machine type D whose operating cost is \$6.00 per hour. Products #3 and #9 use machine type E whose cost is \$7.00. Product #5, which is an aggregate of two outputs, uses both D and E and will be assessed at the rate for type E; the more recent emphasis (observations for 1967) for this conglomerate has been on using type E.

Since the per hour machine operating costs are said (by the firm) not to change according to the utilization rate of the machine, G^M is a null matrix.

$$y = \begin{bmatrix} 6.00 \\ 6.00 \\ 7.00 \\ 6.00 \\ 7.00 \\ 6.00 \\ 6.00 \\ 6.00 \\ 7.00 \\ 6.00 \end{bmatrix} = g^M$$

For the whole period over which the observations were collected the purchase price of the material being used was \$.18 per pound. Since the units considered here are in thousands of pounds, the material is \$180.00 per thousand pounds. The specification of the parameters of Equation 2-4 is

$$r = \begin{bmatrix} 180.00 \\ 180.00 \\ 180.00 \\ 180.00 \\ 180.00 \\ 180.00 \\ 180.00 \\ 180.00 \\ 180.00 \\ 180.00 \end{bmatrix} = g$$

Since the price was fixed throughout the period, G is taken to be a null matrix.

E. Varying the Level of Production

1. The framework

For this application K_p, K_d, and K_c are n by n diagonal matrices. K_0 is an n by 1 vector of constants. The non-zero elements of K_p, K_d, and K_c must be negative as has been shown by Phillips (70) and discussed by Fox, Sengupta, and Thorbecke (34, p. 231). For the i th product: (1) if the ii th element of K_p is zero, the proportional policy is not applied; (2) if the ii th element of K_d is zero, the derivative policy is not applied; (3) if the ii th element of K_c is zero, the integral policy is not applied. That the decision to vary the level of production of each product can be analyzed by the Phillips type function, Equation 2-6, is the hypothesis to be tested empirically in this section.

Recall from the previous description of Equation 2-5

$$W_t = q_t^* + q_t - \overline{q}_t \qquad \text{(Eq. 2-5)}$$

that $W_t = q_{t+1}^*$ if $q_t^* + q_t - \overline{q}_t \geq 0$. Since there was no unfilled demand $W_t = q_{t+1}^*$ for all ten products of this application. Substituting q_{t+1}^* for W_t in Equation 2-6 gives

$$q_{t+1} - q_t = K_0 + K_p(q_{t+1}^* - q_{t+1}^d) + K_d \, \Delta \, (q_{t+1}^* - q_{t+1}^d) + K_c \sum_{j=0}^{N} (q_{t+1-j}^* - q_{t+1-j}^d) \qquad \text{(Eq. 5-11)}$$

where the non-zero elements of K_p, K_d, and K_c are negative since these are diagonal matrices. Attempts will be made to find the best fits of this form for N=2,3,4. N = 4 allows the levels of beginning inventory at times t+1, t, t-1, t-2, and t-3 to affect $q_{t+1} - q_t$. For N = 3, q^*_{t-3} has no affect, and for N = 2, neither q^*_{t-2} nor q^*_{t-3} influences $q_{t+1} - q_t$.

In attempting to estimate the parameters of Equation 5-11, the first issue is that of the well known identification problem since q^*_{t+1} appears within all three independent variables. To estimate the parameters of Equation 5-11, a reduced form must be studied. Letting $\Delta\ q^*_{t+1} = q^*_{t+1} - q^*_t$ and collecting terms leaves

$$q_{t+1} - q_t = K_0 - K_p\, q^d_{t+1} - K_d\, \Delta\, q^d_{t+1} - K_c \sum_{j=0}^{N} q^d_{t+1-j} + (K_p + K_d + K_c)\, q^*_{t+1} + (K_c - K_d)\, q^*_t + K_c \sum_{j=2}^{N} q^*_{t+1-j}$$

It will be assumed that q^d, the target level of inventory, does not change during the months over which data were collected. Therefore,

$$K_d\, \Delta\, q^d_{t+1} = 0$$

$$K_c \sum_{j=0}^{N} q^d_{t+1-j} = K_c\, (N + 1)\, q^d$$

and

$$q_{t+1} - q_t = K_0 - \left[K_p + (N+1)\, K_c\right] q^d + (K_p + K_d + K_c)\, q^*_{t+1} + (K_c - K_d)\, q^*_t + K_c\, (q^*_{t-1} + \ldots + q^*_{t+1-N}). \qquad \text{(Eq. 5-12)}$$

It is Equation 5-12 or

$$q_{t+1} - q_t = a_0 + a_1\, q^*_{t+1} + a_2\, q^*_t + a_3\, (q^*_{t-1} + \ldots + q^*_{t+1-N}) \qquad \text{(Eq. 5-13)}$$

that can be estimated while avoiding a conflict with the identification problem. The correspondence between Equations 5-12 and 5-13 is that

$$a_0 = K_0 - \left[K_p + (N+1)\, K_c\right]\, q^d$$

$$a_1 = K_p + K_d + K_c$$

$$a_2 = K_c - K_d$$

$$a_3 = K_c$$

or

$$K_c = a_3$$

$$K_d = a_3 - a_2$$

$$K_p = a_1 + a_2 - 2\,a_3$$

$$K_0 = a_0 + \left[a_1 + a_2 + (N - 1)\,a_3\right] \quad q^d.$$

The conditions to be fulfilled to obtain negative values for all of the ii th elements of K_p, K_d, and K_c are very stringent.

$$K_p < 0 \quad \text{requires} \quad a_3 < 0$$

$$K_d < 0 \quad \text{requires} \quad a_3 < a_2$$

$$K_p < 0 \quad \text{requires} \quad a_1 + a_2 < 2\,a_3$$

For $a_3 < a_2$, $K_d < 0$, then $a_1 + a_3 < a_1 + a_2$. But $a_1 + a_2 < 2\,a_3$, $K_p < 0$; therefore, $a_1 + a_3 < 2\,a_3$ and $a_1 < a_3$. Then,

$$a_1 < a_3 < 0.$$

A few more conditions on the sizes of the absolute value for a_2 in comparison to a_1 and a_3 might be offered, but the suggestion that there are very few combinations of parameters for $K_p < 0$, $K_d < 0$, $K_c < 0$ should be accepted.

There are six other cases that might occur and in fact will be the ones that appear later in this section. These are the possibilities that less than all three of the proportional, derivative, and integral policies be applied together. For a policy not to be applied its coefficient (K_p, K_d, or K_c, respectively) is zero. The cases follow.

(i) $K_c = 0,\ K_p \neq 0,\ K_d \neq 0.$

$$a_0 = K_0$$

$$a_1 = K_p + K_d$$

$$a_2 = -\ K_d$$

$$K_p = a_1 + a_2$$

$$K_d = -\ a_2$$

$$K_0 = a_0$$

(ii) $K_d = 0,\ K_p \neq 0,\ K_c \neq 0.$

$$a_0 = K_0 - \left[K_p + (N + 1)\ K_c\right]\ q^d$$

$$a_1 = K_p + K_c$$

$$a_2 = a_3 = K_c$$

$$K_p = a_1 - q_2$$

$$K_c = a_2$$

$$K_0 = a_0 + \left[a_1 + N\ a_2\right]\ q^d$$

To estimate this case, Equation 5-13 is replaced by

$$q_{t+1} - q_t = a_0 + a_1 q^*_{t+1} + a_2 (q^*_t + q^*_{t-1} + \ldots + q^*_{t+1-N})$$

(iii) $K_p = 0,\ K_d \neq 0,\ K_c \neq 0$

$$a_0 = K_0 - (N + 1)\ K_c\ q^d$$

$$a_1 = K_d + K_c$$

$$a_2 = K_c - K_d$$

$$a_3 = K_c$$

$$K_c = a_3$$

$$K_d = a_1 - a_3 = a_3 - a_2$$

$$K_0 = a_0 + (N + 1)\ a_3\ q^d$$

which is unlikely to occur.

(iv) $K_p = 0,\ K_d = 0,\ K_c \neq 0.$

$$a_0 = K_0 - (N + 1)\ K_c\ q^d$$

$$a_1 = 0$$

$$a_2 = 0$$

$$a_3 = K_c$$

$$K_0 = a_0 + (N + 1)\ a_3\ q^d$$

$$K_c = a_3$$

(v) $K_p = 0,\ K_c = 0,\ K_d \neq 0.$

$$a_0 = K_0$$

$$a_1 = K_d$$

where the equation to be estimated becomes

$$q_{t+1} - q_t = a_0 + a_1\ (q^*_{t+1} - q^*_t)$$

(vi) $K_c = 0,\ K_d = 0,\ K_p \neq 0.$

$$a_0 = K_0 - K_p\ q^d$$

$$a_1 = K_p$$

$$a_2 = 0$$

$$a_3 = 0$$

$$K_p = a_1$$

$$K_0 = a_0 + a_1 q^d$$

If values for a_1, a_2, and a_3 cannot be found such that $K_p < 0$, $K_d < 0$, and $K_c < 0$, it will be necessary to resort to one of these situations where all of the policies are not applied together. This will be the situation that occurs and is illustrated on the following pages.

2. <u>The estimates</u>

In order to find a set of parameters for each product such that $K_p < 0$, $K_d < 0$, and $K_c < 0$, simple least squares regression was applied to the data for the variables of Equation 5-13 for N = 2,3,4; no estimates that would provide negative K's were obtained. Next, restricted least squares was applied with the constraints being

$$a_3 < 0$$

$$a_3 - a_2 < 0$$

$$a_1 + a_2 - 2\, a_3 < 0$$

for each product; no estimates were found. When restricted least squares is applied, this is always a possibility since a quadratic programming problem is being solved, and the feasible set may be empty if the constraints are stringent.

Such unfortunate results could occur as a result of serial correlation; this is the occurrence of error terms that are not successively independent. This possibility was considered. The error terms were found to be correlated for all ten products. The Durbin Watson d-statistic was found to be larger than 2.0 for all ten cases, and the critical level is less than 1.75 for these applications. The interpretation of this result is that the error terms are negatively correlated. This is what should be expected if $q_{t+1} - q_t$ is to be explained by variables with negative coefficients.

The usual method for dealing with problems having correlated error terms is to apply generalized least squares instead of simple least squares. This technique is well known; Johnston (49) and Goldberger (38) have both described this practice extensively. Both generalized least squares and restricted generalized least squares were applied in the attempt to find the parameters of Equation 5-13 that would be acceptable. $z'y$ and $z'z$ of Equation 5-3 become $z' T'T y$ and $z' T'T z$, respectively, where T is a transformation matrix to account for the correlation among the error terms.

The methods of generalized least squares and restricted generalized least squares also proved ineffective to determine negative K's for Equation 5-13 for all products. For the cases of restricted generalized least squares again there were no estimates. The estimates obtained for the K's using generalized least squares were not acceptable because at least one of the K's was positive in each instance.

Finally, cases (i) through (vi) previously described in this section were considered. For each product some results were obtained. In each instance the regression that gave the most significant a's was selected for the estimated model. This means that if several regressions for one product gave one non-zero a (and therefore K), the one having the highest t-statistic is selected. Thus the selected parameters reject the null hypothesis of the a being zero at the lowest level of significance (smallest Type I error). Such decisions were necessary for products 3, 4, 5, 7, 8, 9, and 10. For product 1 there was only one alternative.

For product 2 the alternatives are those given in Table 4. The values in the parentheses are the t-statistics for the estimates. Of these estimates the alternative where N = 2 was selected since the non-zero estimates had the highest t-statistic to accompany it. For product 6 the alternatives follow. Because of the t-statistic (which is in parentheses as before) the alternative for N = 2 was selected for product 6.

For all of the products the cases selected to enter the model are given in Table 7. The values for q^d were selected for each product so that q^d was the minimum level of inventory that occurred over the months during which data were collected.

$$q_i^d = \min_t q_{it}^* \qquad i = 1,\ldots,n=10$$

The estimates of the parameters for products 5 and 7 are inferior to the others as well as being poor. The significance of the estimates can be judged using the following excerpt from a table for the t-statistic for 16 degrees of freedom. There were eighteen observations remaining after the losses for the number of time periods included in each equation. One may be somewhat concerned by the low value for the R^2 for most of the products; however, one can rely on the significance of the t-statistics since there is a relationship between R^2 and t for a simple linear regression problem with only one independent variable.

$$t = \frac{(R - \rho)^{\frac{1}{2}}\,(n - 2)^{\frac{1}{2}}}{(1 - R^2)^{\frac{1}{2}}}$$

Table 4. Alternative estimates of product 2

N	a_1	a_2	a_3	K_p	K_d	K_c
2	0	0	-.381 (-1.709)	0	0	-.381
3	-.341 (-.892)	0	0	-.340	0	0
3	-.441 (-1.055)	.265 (.659)	0	-.176	-.265	0
4	-.295 (-1.000)	0	0	-.295	0	0
4	-.354 (-1.120)	.241 (.616)	0	-.113	-.241	0

Table 5. Alternative estimates of product 6

N	a_1	a_2	a_3	K_p	K_d	K_c
2	0	0	-.565 (-1.314)	0	0	-.565
3	-.151 (-.240)	0	0	-.151	0	0
3	-.224 (-.319)	0	-.106 (-.271)	-.118	0	-.106

Table 6. Partial t-table

level of significance	.80	.50	.40	.30	.20	.10	.05	.01
t	.258	.689	.862	1.071	1.337	1.746	2.120	2.921

When testing $\rho = 0$, $R^2 = \dfrac{t^2}{t^2 + n - 2}$ which should make the use of the t-statistic a satisfactory criterion on which to judge the estimates.

For product 5 the estimates of $K_c = -.334$ will be used in the model. While an estimate that is significantly different from zero at close to the .30 level of significance is not strongly supported, its use should not be prohibited. For product 7 the estimate $K_d = -.029$ cannot be used; the significance level is above .80. The significance level of the intercept estimate for product 7 is also above .80. Therefore, it shall be accepted that

$$q_{7t+1} - q_{7t} = 0$$

for the period over which the data were gathered. That the differences in the levels of production, $q_{t+1} - q_t$, for products 5 and 7 are hardly explained should not be surprising if you recollect that these were conglomerate items. Evidently the curtailment or reduction in output of one member of the conglomerate is being overcome by the replacement item or additional items being produced.

It is true that several of the intercepts are not significantly different from zero; the significance level of the t-statistic does not approach .05. However, the values for K_0 from Table 7 will be used in the model;

Table 7. Estimates for K_0, K_p, K_d, and K_c

Product	a_0	a_1	a_2	a_3	K_0	K_p	K_d	K_c	N	q^d	R^2
1	277.208 (2.584)	0	0	-1.304 (-2.739)	120.728	0	0	-1.304	2	40	.542
2	93.114 (.713)	0	0	-.381 (-1.709)	47.394	0	0	-.381	2	40	.247
3	130.843 (.880)	-1.942 (-1.440)	0	0	83.492	-1.942	0	0	4	26	.339
4	176.819 (1.464)	0	-1.200 (-3.615)	0	176.819	0	-1.200	0	2	0	.649
5	67.446 (.824)	0	0	-.334 (-.968)	57.446	0	0	-.334	2	15	.223
6	111.206 (.650)	0	0	-.565 (-1.314)	-109.144	0	0	-.565	2	130	.296
7	28.857 (.296)	0	-.766 (-.290)	0	28.857	0	-.029	0	2	32	.070
8	197.772 (2.065)	0	-.766 (-3.126)	0	197.772	0	-.776	0	3	30	.615
9	147.897 (1.800)	0	-.761 (-2.669)	0	147.897	0	-.761	0	3	26	.559
10	200.196 (2.371)	0	-1.406 (-3.636)	0	200.196	0	-1.406	0	2	26	.653

note that the values for K_0 are computed from a_0 and q^d and that the K_0's cannot be statistically judged in a direct manner. The estimated equations that are to become part of the econometric model are summarized below.

$$q_{1t+1} - q_{1t} = 277.208 - 1.304\ (q^*_{1t+1} + q^*_{1t} + q^*_{1t-1})$$

$$q_{2t+1} - q_{2t} = 93.114 - .381\ (q^*_{2t+1} + q^*_{2t} + q^*_{2t-1})$$

$$q_{3t+1} - q_{3t} = 130.843 - 1.942\ q^*_{3t+1}$$

$$q_{4t+1} - q_{4t} = 176.819 - 1.200\ q^*_{4t+1}$$

$$q_{5t+1} - q_{5t} = 67.446 - .334\ (q^*_{5t+1} + q^*_{5t} + q^*_{5t-1})$$

$$q_{6t+1} - q_{6t} = 111.206 - .565\ (q^*_{6t+1} + q^*_{6t} + q^*_{6t-1})$$

$$q_{7t+1} - q_{7t} = 0$$

$$q_{8t+1} - q_{8t} = 197.772 - .766\ (q^*_{8t+1} - q^*_{8t})$$

$$q_{9t+1} - q_{9t} = 147.897 - .761\ (q^*_{9t+1} - q^*_{9t})$$

$$q_{10t+1} - q_{10t} = 200.196 - 1.406\ (q^*_{10t+1} - q^*_{10t})$$

There is one additional set of restrictions that pertains to the inventory levels on the output side. By agreement between the firm and its customers, the maximum and minimum levels of inventory to be maintained for each product are set. Therefore,

$$\min q^*_{it} < q^*_{it} < \max q^*_{it} \text{ for all } i \text{ and } t$$

are another set of constraints to be included in this specific application. These restrictions will play an important role in the following chapter.

F. Analyzing the Purchasing Decision

To estimate the parameters of Equations 2-7 and 2-8 several assumptions will be adopted. First, the relevant values for the lead time must be specified (values of T); second, the number of previous periods that are relevant to a purchasing decision in time t (values of M) needs to be identified. Members of the firm to which the model is being applied suggested that the following values be considered for these parameters.

$M = 2,3$ months

$T = 0,1$ month

Recall from chapter two that the elements of the main diagonal of H_p, H_d, and H_c must be negative. This suggests that restricted least squares may be needed again to find appropriate parameters. Since there is only one material used in the production process, H_p, H_d, and H_c are scalars and must be negative. The set of constraints identified by Equation 5-8 will also be included. However, it will be obvious shortly that finding negative values for H_p, H_d, and H_c will be unlikely enough. Equation 2-8 was substituted into Equation 2-7 to eliminate X^p_{t-T}. The reduced equation to be considered is

$$X_t = b + H_o + (H_c + H_d + H_p)\, X^*_{t-T} + (H_c - H_d)\, X^*_{t-T-1} + H_c \sum_{j=2}^{M} X^*_{t-T-j} - (H_p\, X^d_{t-T} + H_d\, \Delta X^d_{t-T} + H_c \sum_{j=0}^{M} X^d_{t-T-j})$$

This form must be used so that the equation to which least squares is applied is statistically identified.

X^d will be assumed to be a constant. Then $\Delta X^d_{t-T} = 0$ and $\sum_{j=0}^{M} X^d_{t-T-j} = (M+1)X^d$. Therefore,

$$X_t = b + H_o - X^d (H_p + (M+1) H_c) + (H_c + H_d + H_p) X^*_{t-T} +$$

$$(H_c - H_d) X^*_{t-T-1} + H_c \sum_{j=2}^{M} X^*_{t-T-j} \qquad \text{(Eq. 5-14)}$$

Letting

$$B_0 = b + H_o - X^d (H_p + (M+1) H_c)$$

$$B_1 = H_c + H_d + H_p$$

$$B_2 = H_c - H_d$$

and

$$B_3 = H_c$$

makes it possible to replace Equation 5-14 by

$$X_t = B_0 + B_1 X^*_{t-T} + B_2 X^*_{t-T-1} + B_3 \sum_{j=2}^{M} X^*_{t-T-j} .$$

Then

$$H_c = B_3$$

$$H_d = B_3 - B_2$$

$$H_p = B_1 + B_2 - 2 B_3$$

and

$$H_o = B_0 + X^d (B_1 + B_2 + (M-1) B_3) - b$$

The situation here for the H's and the B's is the same as that for the K's and a's, respectively, from the previous section. The properties and restrictions of the B's will not be stated here since they are the same as for the a's with the same subscripts from section E of this chapter.

Attempts were made to estimate Equation 5-14 for all four combinations of T=0,1 and M=2,3. Simple least squares and restricted least squares methods were applied. Only for the case where M = 2 and T = 0 were estimates such that $H_p < 0$, $H_d < 0$, and $H_c < 0$ obtained; for M = 2 and T = 0 both the least squares estimates and restricted least squares estimates gave $H_p < 0$, and $H_d < 0$, and $H_c < 0$, and each technique gave worthy t-statistics, although the restrictions of Equation 5-8 could not be met at $\alpha = .05$ for all of the B's. Since simple least squares estimates will give a smaller sum of squares due to error, this method will be preferred whenever the H's turn out to be negative. The simple least squares estimated relationship follows with the t-statistics being given in parentheses below the parameters.

$$X_t = \underset{(.799)}{721.816} - \underset{(-291.6)}{.524}\, X^*_t + \underset{(108.0)}{.200}\, X^*_{t-1} - \underset{(-51.9)}{.097}\, X^*_{t-2} \qquad \text{(Eq. 5-15)}$$

$$R^2 = .956$$

All of the slope coefficients are statistically significantly different than zero at a 5 per cent significance level. The intercept is significantly different than zero at the 25 per cent significance level. The estimates obtained when restricted least squares was applied were not so significant as those given in Equation 5-15. Since

$B_0 = 721.816$, $B_1 = -.524$, $B_2 = .200$, and $B_3 = -.097$, then

$$H_o = 721.816 - .421\, X^d - b$$

$$H_p = -.297$$

$$H_d = -.130$$

$$H_c = -.097$$

At least one other test statistic needs to be considered for this relationship before one is likely to place much confidence in these parameter estimates. The Durbin-Watson d-statistic must be computed to see if there is an autoregressive process within the structure of Equation 5-14; this is the case if the error terms that are added to Equation 5-14 for statistical estimation are not independent. The Durbin-Watson d-statistic for Equation 5-15 was found to be equal to 1.036. From the Durbin-Watson tables (22, 23) for three independent variables and seventeen observations the upper and lower confidence limits at the 95 per cent level are $d_U = 1.71$ and $d_L = .90$, respectively.

According to Goldberger (38, p. 244) since $d_L \leq d < d_U$ and $4 - d \geq d_U$, the test on positively correlated errors is inconclusive and the hypothesis that the errors are negatively correlated can be rejected, respectively. If the d-statistic clearly implied positively or negatively correlated errors, generalized least squares could be applied to adjust for the interdependence (38, p. 245). With d = 1.036, such an adjustment would hardly be effective.

Suppose that the value of X^d is assumed to be equal to the minimum level of X_t^* within the time horizon for which data were collected; then $X^d = 346.6$. This seems to be a reasonable method for selecting a level for X^d since the variables to which X_t reacts are functions of $X_t^* - X_t^d$, the random variable b and the constant H_o. For $X^d = 346.6$,

$$H_o = 518.226 + b \ .$$

Recall that b is a random variable with a probability density function $f_b(\cdot)$ that has yet to be specified. It has already been argued that $b < 0$, $b = 0$, and $b > 0$ all have meaningful interpretations. The density function for each element of b cannot be stipulated until the relevant lead time, T, is determined. Whenever T = 0 for a material, the element of b for that material will always be zero and will not be a random variable. This must be the case since T equal to zero implies that the quantity ordered in period t is received in that same period. For the application of this chapter, the scalar $H_o = 518.226$.

b = 0 will not always be the case and so a method for estimating the i th element of H_o should be specified for $b_i \neq 0$. It will be assumed that the expected value of b, E(b), is known or estimated as the mean of observations on $(X_t - X^p_{t-T})$. Let $E(b) = \bar{b}$. If the simple least squares estimates for Equation 5-14 provide $H_p < 0$, $H_d < 0$, and $H_c < 0$ for the relevant material, the parameter estimates for the identified form, Equation 5-12, give unbiased estimates for B_0, B_1, B_2, and B_3; let these estimates be denoted by $\bar{B}_0$, $\bar{B}_1$, $\bar{B}_2$, and $\bar{B}_3$, respectively.

From the expressions that immediately follow Equation 5-14

$$b = B_0 + X^d\ (B_1 + B_2 + (M-1)B_3) - H_o \qquad \text{(Eq. 5-16)}$$

which implies that

$$E(b) = E(B_0) + X^d \left\{ E(B_1) + E(B_2) + (M-1)\ E(B_3) \right\} - H_o.$$

Note that the random variable b is a linear combination of the B's. Whenever it is realistic to assume that the error term of the estimated equation is normally distributed, the B's, and therefore b, will be normally distributed. Taking the expected value of Equation 5-16 and transposing the terms gives

$$H_o = E(B_0) + X^d \left\{ E(B_1) + E(B_2) + (M-1)\ E(B_3) \right\} - E(b)$$

If restricted least squares had been needed to find $H_p < 0$, $H_d < 0$, and $H_c < 0$, the parameter estimates for Equation 5-15 will be biased estimates of the expected values. This should be clear from the derivation of the bias of the restricted least squares estimates that was derived at the conclusion of section B of this chapter.

Provided that b is a normally distributed random variable, its density function is completely determined by the expected value, E(b), and the variance, Var(b).

$$\mathrm{Var}(b) = \mathrm{Var}\ (B_0 + X^d \left\{B_1 + B_2 + (M-1)B_3 - H_o)\right\}$$

$$= \mathrm{Var}\ (B_0) + X^d \left\{\mathrm{Var}(B_1) + \mathrm{Var}(B_2) + (M-1)^2\ \mathrm{Var}(B_3) + 2\ \mathrm{COV}(B_1B_2) + 2(M-1)\ \mathrm{COV}(B_1B_3) + 2(M-1)\ \mathrm{COV}(B_2B_3)\right\}$$

If the B's are assumed to be independent, the covariances are zero and

$$\mathrm{Var}(b) = \mathrm{Var}(B_0) + X^d\ \mathrm{Var}(B_1) + \mathrm{Var}(B_2) + (M-1)^2\ \mathrm{Var}(B_3)$$

Therefore, the form of $f_b(\cdot)$ is determined for $T \neq 0$.

In summary, the parameters determined in this section are:

$H_o = 518.226$

$H_p = -.097$

$H_d = -.297$

$H_c = -.130$

$X^d = 346.6$

$M = 2$

$T = 0$

and

$b = 0.$

G. The Demand and Revenue Structure

In chapter two it was shown that either

$$\text{(i)} \quad q^*_{it+1} \geq 0, \; q^u_t = 0, \text{ and } \bar{q}_t = q^s_t$$

or

$$\text{(ii)} \quad q^*_{it+1} = 0, \; q^u_t > 0, \; \bar{q}_t > q^s_t \, .$$

Therefore,

$$\bar{q}_t = q^s_t + q^u_t \qquad \text{(Eq. 5-17)}$$

The situation can be summarized by the following probability statements.

$$\Pr(\bar{q}_t > q^s_t) + \Pr(q^u_t > 0) = \alpha \qquad \text{(Eq. 5-18)}$$

and

$$\Pr(\bar{q}_t = q^s_t) = \Pr(q^u_t = 0) = 1 - \alpha \qquad \text{(Eq. 5-19)}$$

where α should be quite small. Obviously, the greater the potential loss from incurring unfilled demand, the closer α will be to zero.

To estimate the demand functions of Equation 2-10, one would like to find the elements of the vector s and the matrix S by minimizing the squared error, e'e, of

$$p = s + S\,\bar{q} + e \qquad \text{(Eq. 5-20)}$$

Unfortunately, it is rarely the case that a firm maintains data on the quantity demanded. This is a result of the usual decision to hold inventory of each product so that $q^*_{t+1} > 0$, and case (i) of the first paragraph of this section holds. At this point one might be inclined to replace $\bar{q}$ by q^s in Equation 5-20 and proceed to estimate the relationship using observations on p and q^s. But, no matter how small α may be, the fact that α is not always zero in Equations 5-18 and 5-19 must be considered.

If observations on q^s are to be used as a proxy for $\bar{q}$, the relationship that is really being estimated is

$$p = s + S\ (q^s + q^u) + e \qquad \text{(Eq. 5-21)}$$

or

$$p = s + S\ q^s + (S\ q^u + e). \qquad \text{(Eq. 5-22)}$$

To estimate Equation 5-22 is not so simple as it may appear unless q^u is zero for all observations ($\alpha = 0$). For $\alpha \neq 0$, applying least squares regression to the variables p and q^s exemplifies the case where there are errors in the variables (49, Ch. 6). This results since the error term is now $S\ q^s + e$ instead of simply e. The parameter estimates for S will be both biased and inconsistent if simple least squares were to be applied to Equation 5-22. For least squares estimates of S to be meaningful, observations must be determined for q^u and the variable $q^s + q^u$ must be used as the independent variable. Johnston (49) gives extensive coverage to estimation problems involving errors in the variables, which are also called errors in measurement by some econometricians.

The estimation of the demand functions for the application of this chapter is somewhat simplified because only case one of the first paragraph of this section is relevant. For all of the months over which the data were collected, there was no unfilled demand for any product. For this application $\alpha = 0$. Therefore, simple least squares can be applied to Equation 5-22 and the usual properties (best, linear, unbiased estimates) of the estimates will hold. In other words $q^u = 0$ for all cases and regressing p on q^s will not involve errors in the variables. The estimates

Table 8. Parameters of the demand functions

Product	s_i	S_{ii}	R^2
1	76.104 (50.267)*	.006 (.759)	.176
2	62.707 (78.384)*	-.004 (-.808)	.187
3	117.532 (58.766)*	-.018 (-.845)	.195
4	65.930 (23.297)*	-.022 (-1.803)*	.382
5	60.711 (37.239)*	-.024 (-6.460)*	.865
6	88.767 (6.924)*	-.023 (-.497)	.113
7	110.469 (25.395)*	-.118 (-12.422)*	.944
8	78.521 (98.167)*	-.0003 (-.083)	.019
9	203.084 (15.828)*	-.113 (-.838)	.189
10	162.028 (6.137)*	-.888 (-5.390)*	.778

*Significant at the .05 level.

for the elements of s and S for products one through ten are given in Table 8. The t-statistics appear in parentheses below each estimate.

From the t-statistics in Table 8 it can be argued that products 1,2,3, 6,8, and 9 are sold at nearly fixed prices; the foundation of this is that $\frac{dp}{dq}s$ is not significantly different from zero as shown by the t-statistics.

In section E of this chapter the relationship between the t-statistics and R^2 was illustrated. This should make the use of the t-statistic satisfactory in determining which products are sold at fixed prices. The demand relations to be used in the programming model of chapter six will be as follows.

product	relation
1	$p_1 = 76.104$
2	$p_2 = 62.707$
3	$p_3 = 117.532$
4	$p_4 = 65.930 - .022\ q_4^s$
5	$p_5 = 60.711 - .034\ q_5^s$
6	$p_6 = 88.767$
7	$p_7 = 110.469 - .118\ q_7^s$
8	$p_8 = 78.521$
9	$p_9 = 203.084$
10	$p_{10} = 162.028 - .888\ q_{10}^s$

The signs of the slope coefficients for products 4, 5, 7, and 10 agree with the premise from economic theory that if $\frac{dp}{dq^s} \neq 0$, then $\frac{dp}{dq^s} < 0$.

In the matrix formulation of Equation 2-10 s is a vector and S is a diagonal matrix.

$$p = \begin{bmatrix} 76.104 \\ 62.707 \\ 117.532 \\ 65.930 \\ 60.711 \\ 88.767 \\ 110.469 \\ 78.521 \\ 203.084 \\ 162.028 \end{bmatrix} + \begin{bmatrix} 0 & 0 & & & & & & & & & \cdots & & 0 \\ 0 & 0 & & & & & & & & & \cdots & & \\ 0 & 0 & 0 & 0 & & & & & & & \cdots & & . \\ . & \cdots & 0 & -.022 & & 0 & & & & & \cdots & & \\ & & & & 0 & -.034 & & 0 & & & \cdots & & . \\ . & \cdots & & & & & 0 & 0 & 0 & & \cdots & & \\ & & & & & & & 0 & -.118 & 0 & \cdots & & . \\ . & \cdots & & & & & & & 0 & 0 & 0 & \cdots & \\ & & & & & & & & & 0 & 0 & 0 & 0 \\ 0 & 0 & . & . & . & & & & & & 0 & & -.888 \end{bmatrix} q^s$$

Several times in this study the point has been made that a constant price need not imply that these items are sold in markets that are perfectly competitive.

The firm's total revenue is the scalar $q^{s\prime}\ p$ where

$$q^{s\prime}\ p = \sum_{i=1}^{10} q_i^s p_i = 76.104\ q_1^s + 62.707\ q_2^s + 117.532\ q_3^s + 65.930\ q_4^s + 60.711\ q_5^s + 88.767\ q_6^s + 110.469\ q_7^s + 78.521\ q_8^s + 203.084\ q_9^s + 162.028\ q_{10}^s - .022\ (q_4^s)^2 - .034\ (q_5^s)^2 - .118\ (q_7^s)^2 - .888\ (q_{10}^s)^2$$

The last commitment towards estimating the firm's revenue and demand structure is to determine the probability density function for each of the elements of $\overline{q}$. Since $q^u = 0$ and $\alpha = 0$ for this application, the density function for $\overline{q}_i$ is the same as that for q_i^s. If this were not the case, the density function for $\overline{q}_i$ could be determined by fitting the density function to $q_i^s + q_i^u$ instead of q_i^s.

There are several approaches that might be utilized to fit frequency curves to the data on the q_i^s's. Those that are among the best known have been summarized by Professor Kendall in The Advanced Theory of Statistics (50, Ch. 6). In this study the data collected on each q_i^s will be fitted to one of the density functions from the family of the Pearson curves or distributions. Many of the well known density functions that are studied in modern statistics are found among those in the family of Pearson curves. Furthermore, several density functions that are not strictly among those in the Pearson family can be closely approximated by one or more of the Pearson family. Examples can be found in the exposition by W. P. Elderton (26) where the Pearson curves are studied extensively.

The Pearson distributions are represented by the alternative solutions to the following differential equation.

$$\frac{dy}{dz} = \frac{(z - a)\ y}{b_0 + b_1\ z + b_2\ z^2}$$

which can be transformed to

$$\frac{d(\log y)}{dv} = \frac{v}{B_0 + B_1\ v + B_2\ v^2}$$

by letting

$$v = z - a$$

$$B_0 = b_0 + b_1\ a + b_2\ a^2$$

$$B_1 = b_1 + 2\ b_2\ a$$

$$B_2 = b_2$$

The values for a, b_0, b_1, and b_2 are determined by the first four moments of the data that are collected on the random variable z. The solutions and therefore the particular type of curve from the Pearson family is determined by the roots of the following quadratic equation.

$$B_0 + B_1\ v + B_2\ v^2.$$

These types of curves along with examples are delineated by Kendall (50) and Elderton (26).

The roots of $B_o + B_1\ v + B_2\ v^2$ were found to be real and have opposite signs for the data on q^s for all products except number nine. Real roots of opposite sign for the quadratic equation given above is the condition for a Type I Pearson distribution whose density function is

$$f(v) = k\, v^{m_1} \left[1 - \left(\frac{v - a_1}{a_2}\right)\right]^{m_2} \qquad 0 \leq v \leq a_1 + a_2$$

$$m_1 > -1, \; m_2 > -1$$

$$k = \frac{(a_2)^{m_1} \, (m_1 + m_2 + 1)!}{(a_1 + a_2)^{m_1 + m_2} \, m_1 ! \, m_2 !}$$

In this formulation v represents q_i^s when the i th product is being considered. Note that $i \neq 9$. The factorials can be evaluated using Stirling's approximation that $u! = (2\pi u)^{\frac{1}{2}} \left(\frac{u}{e}\right)^u$. The estimates for the parameters of the density functions of q_i^s for i=1,2,3,4,5,6,7,8, and 10 follow in Table 9.

For product nine the member of the Pearson family that fits the data is Type III. The fit is quite good. This is shown by the fact that a Type III curve requires $B_2 = 0$, and for product nine $B_2 = -.0002$. The density function and parameters for q_9^s are:

$$f(v) = k \left(1 - \frac{v}{a}\right)^g e^{cv} \qquad 0 \leq v \leq a$$

$$k = \frac{1}{a(g-1)!} \left(\frac{g}{e}\right)^g \qquad c = -\frac{g}{a}$$

where

a = 119.942

g = 34.233

c = -.285

mean = 58.952

standard deviation = 20.797

This concludes the specification of the density functions and para-

Table 9. Parameters of density functions 1-8, 10

Product	1	2	3	4	5	6	7	8	10
m_1	-.081	13.436	1.859	-.223	-.487	2.150	-.838	-.210	-.297
m_2	1.326	2.112	2.205	1.357	.520	1.068	-.089	-.016	.689
a_1	-12.090	380.473	51.581	-47.650	-4627.33	203.70	363.468	146.344	-106.91
a_2	198.911	59.816	61.174	290.205	4939.87	101.165	38.644	11.215	248.373
$a_1 + a_2$	186.821	440.289	112.755	242.555	312.54	304.861	402.113	157.559	141.468
mean	81.250	74.000	62.000	99.762	87.500	200.000	101.895	167.143	48.524
standard deviation	40.853	39.050	21.178	51.511	77.973	59.802	100.055	47.018	34.991

meters for the quantity demanded in this application. Again it should be pointed out that the quantity demanded and shipped are the same since there was no unfilled demand; when this is not the case, the levels of the q^u's can be calculated and the density functions and demand functions can be fitted to $q^s + q^u$.

H. Coefficients of the Objective Functions

Finally the remaining parameters of the cost function must be specified. Estimates for $\bar{c}$, $\hat{c}$, and c^* are needed. These are the coefficients that appear in Equation 3-2. Since $\hat{c}$ is a column vector of n elements and $\bar{c}$ is an n by n diagonal matrix, there are 2n parameters relevant to output inventory to be specified. In chapter three a method was devised to estimate $\bar{c}_{ii}$ and $\hat{c}_i$ for i = 1, ..., n. It was suggested that reasonable estimates could be obtained using

$$\bar{c}_{ii} = \frac{c_i + c_i^u}{q_{it+1}^* + q_{it}^u} \qquad i=1,\ldots,n$$

and

$$c_i = \frac{c_i^u q_{it+1}^* - c_i q_{it}^u}{q_{it+1}^* + q_{it}^u} \qquad i=1,\ldots,n$$

where

c_i^u is the per unit loss incurred from the occurrence of unfilled demand, and

c_i is the per unit cost of holding finished goods inventory for one period.

For this application, $q_{it}^u = 0$ for all i and t. Therefore,

$$\bar{c}_{ii} = \frac{c_i + c_i^u}{q^*_{it+1}} \quad \text{and } \hat{c}_i = c_i^u \text{ for } i=1,\ldots,n.$$

Estimates for the c_i's were obtained using a formula given by the firm to which the application is attempted. The number of square feet needed to store one thousand items of product i is $\emptyset_i$, where

$$\emptyset_i = \frac{(\text{ozs. capacity of the product} + 4)(2)}{5}$$

This method for estimating the $\emptyset_i$'s is highly empirical, but, as is so often the case, is an approach which the firm finds reasonably reliable.

Given the estimates for the $\emptyset_i$'s, the c_i's are determined to be the allocated cost per year per square foot divided by twelve. The annual cost per square foot in the company warehouse is \$.45 per square foot. The monthly cost is \$.0375. Therefore, $c_i = \emptyset_i$ (.0375) for $i=1,\ldots,n$.

$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \\ c_7 \\ c_8 \\ c_9 \\ c_{10} \end{bmatrix} = \begin{bmatrix} 14.4 \\ 10.4 \\ 27.2 \\ 10.4 \\ 15.2 \\ 14.8 \\ 14.4 \\ 14.4 \\ 52.8 \\ 27.2 \end{bmatrix} \cdot \begin{bmatrix} .0375 \end{bmatrix} = \begin{bmatrix} .540 \\ .490 \\ 1.020 \\ .490 \\ .570 \\ .555 \\ .540 \\ .540 \\ 1.980 \\ 1.020 \end{bmatrix}$$

To obtain values for the c_i^u's one must try to measure the per unit loss from the firm's not being able to fill the customer's request. If the firm will be able to fill the order in τ periods, and a relevant monthly discount rate is denoted by r, the firm's value of the per unit price to be received declines from p to p^* where

$$p^* = \frac{p}{(1 + r)^{\tau}}$$

The loss incurred by the firm from delivering the items at time τ instead of at the time of the demand is $p - p^*$, where

$$p - p^* = p\left[1 - \frac{1}{(1 + r)^{\tau}}\right]$$

This will be used as an approximate value for c_u--the per unit cost of having unfilled demand for τ periods.

It will be assumed that the relevant monthly discount rate is 1 per cent and that any unfilled demand is filled during the following month. Then

$$p - p^* = p\left[1 - \frac{1}{1.01}\right] = .01\ p$$

To estimate p in this formulation, the elements of the vector s will be used. To include the additional term of the demand function, $S\ \overline{q}$, would hardly influence the estimates for the c_i^u's since the non-zero elements of S are quite small and would nearly disappear when multiplied by the coefficient of p in the estimate for $c^u = p - p^*$. The estimates for the c_i^u's are:

$$\begin{bmatrix} c_1^u \\ c_2^u \\ c_3^u \\ c_4^u \\ c_5^u \end{bmatrix} = \begin{bmatrix} 76.104 \\ 62.707 \\ 117.532 \\ 65.930 \\ 60.711 \end{bmatrix} [.01] = \begin{bmatrix} .761 \\ .627 \\ 1.175 \\ .659 \\ .607 \end{bmatrix}$$

$$\begin{bmatrix} c_6^u \\ c_7^u \\ c_8^u \\ c_9^u \\ c_{10}^u \end{bmatrix} \quad \begin{bmatrix} 88.767 \\ 110.469 \\ 78.521 \\ 203.084 \\ 162.028 \end{bmatrix} \qquad \begin{bmatrix} .888 \\ 1.105 \\ .785 \\ 2.031 \\ 1.620 \end{bmatrix}.$$

One may be somewhat reluctant to have used the elements of s to estimate the elements of c^u. However, it should be recognized that the per unit losses from not having the item available depend on the revenue that would have to be derived from selling the item.

Using these estimates for the c_i^u's, the parameters $\hat{c}_i$, i=1,...,n are determined since

$$\hat{c}_i = c_i^u \text{ when } q_i^u = 0 \qquad \text{for } i=1,\ldots,n.$$

Since $q_i^u = 0$, $\bar{c}_{ii} = \dfrac{c_i + c_i^u}{q^*_{it+1}}$. Rather than revise the estimates for the output, one value for q^*_{it+1} will be selected for the period over which data were collected. As a representative value the average beginning inventory for each product will be used. The estimates and computations for finding the $\bar{c}_{ii}$'s follow in Table 10.

The remaining coefficient to be specified for Equation 3-2 is c^* which is a scalar. The information received from the firm concerning raw material storage was that "a figure of 20% of raw material cost is assumed, and includes obsolescence, pilferage, and all other charges". In section D of this chapter the raw material cost was shown to be $180.00 per thousand

Table 10. Estimates of cost coefficients

Product i	c_i	$c_i^u = \hat{c}_i$	mean of q^*_{it+1}	$\bar{c}_{ii}$
1	.540	.761	182.544	.007
2	.490	.627	217.283	.005
3	1.020	1.175	51.194	.043
4	.490	.659	148.630	.008
5	.570	.607	189.720	.006
6	.555	.888	219.060	.007
7	.540	1.105	363.364	.005
8	.540	.785	233.961	.006
9	1.980	2.031	205.400	.019
10	1.020	1.620	123.061	.021

pounds. Therefore, the cost of storing a thousand pounds of the raw material for one year would be (20% ($180.00) = $36.00 or $3.00 per month. This will be the estimate for c^*.

$$c^* = 3.00$$

This concludes the estimation of the parameters of the model. A summary follows.

I. Summary

It is an extensive task to summarize the estimates of the model which have been consummated in this chapter. This will be completed here in as brief a fashion as possible; it must be done since the optimization problems of the following chapter will be applications of the techniques delineated in chapter four to this estimated model.

$$q_{1t} = 1.000 + 2.269\ x^L_{1t} + .121\ x^M_{1t} + 1.002\ \bar{x}_{1t}$$

$$q_{2t} = 1.000 + 4.167\ x^L_{2t} + .254\ x^M_{2t} + .256\ \bar{x}_{2t}$$

$$q_{3t} = 1.000 + 3.167\ x^L_{3t} + .170\ x^M_{3t} + .738\ \bar{x}_{3t}$$

$$q_{4t} = 31.680 + 1.962\ x^L_{4t} + .202\ x^M_{4t} + .737\ \bar{x}_{4t}$$

$$q_{5t} = 15.431 + 5.921\ x^L_{5t} + .098\ x^M_{5t} + .461\ \bar{x}_{5t}$$

$$q_{6t} = 1.000 + 4.473\ x^L_{6t} + .193\ x^M_{6t} + .287\ \bar{x}_{6t}$$

$$q_{7t} = 1.000 + 0.100\ x^L_{7t} + .599\ x^M_{7t} + .050\ \bar{x}_{7t}$$

$$q_{8t} = 31.830 + 0.100\ x^L_{8t} + .010\ x^M_{8t} + 1.290\ \bar{x}_{8t}$$

$$q_{9t} = 1.000 + 4.251\ x^L_{9t} + .111\ x^M_{9t} + .358\ \bar{x}_{9t}$$

$$q_{10t} = 1.000 + 5.375\ x^L_{10t} + .194\ x^M_{10t} + .0001\bar{x}_{10t}$$

$$q_{1t+1} - q_{1t} = 277.208 - 1.304\ (q^*_{1t+1} + q^*_{1t} + q^*_{1t-1})$$

$$q_{2t+1} - q_{2t} = 93.114 - .381\ (q^*_{2t+1} + q^*_{2t} + q^*_{2t-1})$$

$$q_{3t+1} - q_{3t} = 130.843 - 1.942\ q^*_{3t+1}$$

$$q_{4t+1} - q_{4t} = 176.819 - 1.200\ q^*_{4t+1}$$

$$q_{5t+1} - q_{5t} = 67.446 - .334\ (q^*_{5t+1} + q^*_{5t} + q^*_{5t-1})$$

$$q_{6t+1} - q_{6t} = 111.206 - .565\ (q^*_{6t+1} + q^*_{6t} + q^*_{6t-1})$$

$$q_{7t+1} - q_{7t} = 0$$

$$q_{8t+1} - q_{8t} = 197.772 - .766\ (q^*_{8t+1} - q^*_{8t})$$

$$q_{9t+1} - q_{9t} = 147.897 - .761\ (q^*_{9t+1} - q^*_{9t})$$

$$q_{10t+1} - q_{10t} = 200.196 - 1.406\ (q^*_{10t+1} - q^*_{10t})$$

$$W_{it} = q^*_{it} + q_{it} - \overline{q}_{it} \qquad i = 1,\ldots,10$$

$$\min q^*_{it} < q^*_{it} < \max q^*_{it} \text{ for all } i \text{ and } t$$

$$X_t = 721.816 - .524\ X^*_t + .200\ X^*_{t-1} - .097\ X^*_{t-2}$$

$$X^*_{t+1} = X^*_t + X_t - \overline{X}_t$$

$$X_t = x_{1t} + \ . \ . \ . + x_{10t}$$

$$\overline{X}_t = \overline{x}_{1t} + \ . \ . \ . + \overline{x}_{10t}$$

$$\begin{aligned} TIC_t = {} & .007\ W^2_{1t} + .005\ W^2_{2t} + .043\ W^2_{3t} + .008\ W^2_{4t} + .006\ W^2_{5t} + \\ & .007\ W^2_{6t} + .005\ W^2_{7t} + .006\ W^2_{8t} + .019\ W^2_{9t} + .021\ W^2_{10t} - \\ & .761\ W_{1t} - .627\ W_{2t} - 1.175\ W_{3t} - .659\ W_{4t} - .607\ W_{5t} - \\ & .888\ W_{6t} - 1.105\ W_{7t} - .785\ W_{8t} - 2.031\ W_{9t} - 1.620\ W_{10t} + \\ & 3.00\ (X^*_t - X^*_{t-1}) \end{aligned}$$

$$\begin{aligned} TPC_t = {} & 180.00\ x_{1t} + 180.00\ x_{2t} + 180.00\ x_{3t} + 180.00\ x_{4t} + \\ & 180.00\ x_{5t} + 180.00\ x_{6t} + 180.00\ x_{7t} + 180.00\ x_{8t} + \\ & 180.00\ x_{9t} + 180.00\ x_{10t} + 16.16\ x^L_{1t} + 16.16\ x^L_{2t} + \\ & 18.12\ x^L_{3t} + 16.16\ x^L_{4t} + 16.16\ x^L_{5t} + 16.16\ x^L_{6t} + 16.16\ x^L_{7t} + \\ & 16.16\ x^L_{8t} + 16.16\ x^L_{9t} + 16.16\ x^L_{10t} + 6.00\ x^M_{1t} + 6.00\ x^M_{2t} + \\ & 7.00\ x^M_{3t} + 6.00\ x^M_{4t} + 7.00\ x^M_{5t} + 6.00\ x^M_{6t} + 6.00\ x^M_{7t} + \\ & 6.00\ x^M_{8t} + 7.00\ x^M_{9t} + 6.00\ x^M_{10t} \end{aligned}$$

$$TR = 76.104\ q^s_{1t} + 62.707\ q^s_{2t} + 117.532\ q^s_{3t} + 65.930\ q^s_{4t} +$$

$$60.711\ q^s_{5t} + 88.767\ q^s_{6t} + 110.469\ q^s_{8t} + 203.084\ q^s_{9t} +$$

$$162.028\ q^s_{10t} - .022\ (q^s_{4t})^2 - .034\ (q^s_{5t})^2 - .118\ (q^s_{7t})^2 -$$

$$.888\ (q^s_{10t})^2$$

$f_1(q^s_1)$, $f_2(q^s_2)$, $f_3(q^s_3)$, $f_4(q^s_4)$, $f_5(q^s_5)$, $f_6(q^s_6)$, $f_7(q^s_7)$, $f_8(q^s_8)$,

and

$f_{10}(q^s_{10})$ are Pearson Type I curves.

$f_9\ (q^s_9)$ is a Pearson Type III curve.

$P_t = TR_t - TPC_t - TIC_t - TFC_t$

$TVC_t = TPC_t + TIC_t$

Note that to optimize P_t is to optimize P^*_t where

$P^*_t = TR_t - TPC_t - TIC_t$

since TFC_t is a constant.

VI. THE OPTIMIZATION PROBLEMS

A. Introduction

Finally, two applications of the estimated model of chapter five can be offered. The model has over one hundred variables and nearly sixty constraints for each time period that one may choose to study. Unfortunately, the determination of optimal activity levels is highly sensitive to the size of the available computational equipment. It should be noted that the needs for solving large quadratic programming problems far exceed those for solving similar linear programming problems.

These considerations very much affected the assumptions made in section B of this chapter. Many of these suppositions represent major simplifications of the applications that were outlined in chapter three. In section C, solutions to the profit maximization and cost minimization problems are provided for several time periods. These results will then be compared to the actual costs, revenues, and profits of the firm for the same levels of production and input utilization.

B. The Framework of the Solutions

The framework of both the cost minimization and profit maximization problems will be that of simple quadratic programming problems over time. For periods prior to the one for which a solution is sought, values of the variables must be known or assumed. The problem is to find the allocations for one period, t. Then these values may be accepted as given data for the decision in period t+1 and those that follow. Thus, the problems that are being solved are a series of two stage programming problems.

Suppose that decisions are to be made for the time period t. It shall be presumed that beginning inventories of the finished goods and materials are given. Therefore, for the i th output, $q^*_{i,\ t-j}$, i=1, ..., 10 and j=0, 1, ..., t, as well as X^*_{t-j} j=0, 1, ..., t are known. This leaves q^s_{it} , q_{it}, x^L_{it}, x^M_{it}, $\bar{x}_{it}$ and q^*_{it+1} to be determined for i=1, ..., 10. Also X_t must be computed. Given X^*_t, summing $\bar{x}_{it}$ over i, and computing X_t will determine the beginning inventory for period t+1. To emphasize that decisions are only being determined within the period t, it should be recognized that in selecting q^*_{it+1} and X^*_{t+i}, the ending inventories for period t are being specified.

For period t it would be desirable to obtain optimal allocations and dollar values that are comparable to the actual events experienced within the firm. Therefore, the values for q_{it}, x^L_{it}, x^M_{it}, and $\bar{x}_{it}$ will be fixed for the t th period for all ten products. t will range from one to four and those data are taken from the actual records that were made available for the first four periods. Given this information, the maximum profits and minimum costs are to be determined. Then these values can be compared to the actual costs and revenues computed from the data of the firm.

Assuming values for the q_{it}'s and q^*_{it}'s, leaves the q^s_{it}'s and q^*_{it}'s to be found. Since $q^u_{it} = 0$ for this application,

$$q^s_{it} = q^*_{it} + q_{it} - q^*_{it+1}$$

and substitutions can be made for each q^s_{it} in terms of q^*_{it+1}. Only the one set of variables remain to be directly determined on the output side. These are the ending inventories of period t or the beginning inventories of period t+1. From these the quantities that were shipped in period t can be computed.

Given the $\overline{x}_{it}$'s, $\overline{X}_t$ is known. This leaves X^*_{t+1} and X_t to be determined using X^*_t and $\overline{X}_t$. Since,

$$X^*_{t+1} = X^*_t + X_t - \overline{X}_t$$

finding either X^*_{t+1} or X_t will determine the other one. From the estimated model X_t is computed via

$$X_t = 721.816 - .524\ X^*_t + .200\ X^*_{t-1} - .097\ X^*_{t-2}$$

then X^*_{t+1} is also known.

These assumptions and vast simplifications appear to reduce the two problems to optimization problems having only ten variables from the model for any one period: q^*_{it+1}, i=1, ..., 10. Recall from the previous chapter that upper and lower bounds on the finished goods inventory levels are fixed by the corporate-purchaser agreements. The restrictions for the programming problems are

$$L_i < q^*_{it+1} < U_i \qquad i=1, \ldots, 10$$

These inequalities provide twenty constraints and thirty variables. Twenty of the variables are slack variables which represent the deviations of q^*_{it+1} from L_i and U_i. The slack variables themselves are not unrelated since a value for $q_{it+1} - L_i$ will imply a value for $q^*_{it+1} - U_i$, given U_i and L_i.

That there have been many limiting assumptions to this point is clear. However, there are two contributions that these presumptions may offer. First, the remaining optimization problems have been reduced to relatively small ones. Second, for several time periods the optimal revenues, costs, and profits for the given levels of q_{it}, x^L_{it}, x^M_{it}, and $\overline{x}_{it}$ i=1, ..., 10 can be compared with the actual dollar values.

C. The Results

All figures for costs, revenues, and profits discussed in this section are in thousands of dollars. In Table 11 the actual levels of profits, P^*, revenues, TR, and costs (production, TPC and inventory, TIC) are given for the first four months over which data were collected. These figures were estimated from the records offered by the firm.

The maximum levels of profits for the same four month period appear in Table 12. For these maxima the relevant costs and revenues are also given. For identification purposes, the profits, total revenue, total cost, total production, and total inventory cost in Table 12 will be signified by P^{*o}, TR^o, TC^o, TPC^o, and TIC^o, respectively. These are the solutions to problem one that was described in chapter three.

For each period the maximum level of profits exceeds the actual level given in Table 11. Also the optimal levels of TC, TPC, and TIC are below their respective actual values. These facts lead to several questions that must now be answered.

Can it be assured that the values for P^{*o} are maximum levels of profits? Recall that q_t, q^*_t, x^L_t, x^M_t, and $\overline{x}_t$ have been taken to be predetermined constants for period t. Suppose that

$$q^*_{t+1} - k = - q^s_t$$

then

Table 11. Actual values

t	P^*	TR	TC	TPC	TIC
1	158.7	273.9	115.2	115.2	.018
2	126.5	228.1	101.6	101.5	.058
3	109.5	237.1	127.6	127.5	.105
4	71.5	191.4	119.9	119.5	.405

Table 12. Optimal values

t	P^{*o}	TR^o	TC^o	TPC^o	TIC^o
1	380.2	479.8	99.6	99.6	.003
2	158.4	270.1	111.7	111.7	.003
3	117.7	227.5	109.8	109.8	.004
4	116.9	232.0	115.1	115.1	.003

$$(q_{t+1}^*)^2 - 2\,k\,q_{t+1}^* + k^2 = (q_t^s)^2 \ .$$

The quadratic terms of $P^* = TR - TPC - TIC$ can be manipulated to involve only $(q_{t+1}^*)^2$ with negative coefficients. Therefore, further increases of q_{t+1}^* above its minimum acceptable level will reduce profits. Likewise, this is the affect of a reduction in q_t^s. The lowest permissable level of finished goods inventory is $q_{t+1}^* = h$, and $q_t^s = q_t^* + q_t - h$.

This discussion is the verbal interpretation of what is implied when it is said that the second order mathematical conditions hold for a non-linear optimization problem. Consequences of the conditions are that any change in the level of the decision variable will reduce the value of the objective function being maximized or violate a constraint. The economic implication here is that the change in profits for an increase in q_{it+1}^* or a reduction in q_{it}^s would be negative. The maximum levels of profits for the various time periods occur where q_{t+1}^* is at its lower bound and $q_t^s = q_t^* + q_t - h$. Since q_t^* and q_t are assumed for the first period, if $q_t^* > h$, then $q_t^s = q_t + q_t^* - h > q_t$. However, as soon as possible a regular pattern will evolve with $q_t^s = q_t$ and $q_{t+1}^* = q_t^* = h$.

A second result that appears in Table 12 is that the levels of TC^o, TPC^o, and TIC^o for the profit maximization problem are themselves the minimum levels of the costs and the solutions to the second problem that was delineated in chapter three. This occurs because the coefficients of the quadratic terms of the total cost function are positive and the squared variables are the elements of q_{t+1}^*. Therefore, any increase in q_{t+1}^* will raise the total cost. This is because the sign of any of the partial derivates of TC with respect to q^*_{t+1} is positive and multiplied by q^*_{t+1}. Therefore, any increase in q^*_{t+1} above its minimum boundary, h, will increase TC and TIC. Again, this is the interpretation of the second order condition of the quadratic optimization problem. The activity levels of the variables determined for the profit maximization and cost minimization problems are the same. As soon as possible, $q^s_t = q_t$ and $q_{t+1} = q^*_t = h$.

Comparing the figures from Tables 11 and 12 shows that the maximum profits exceed the actual profits, and minimum costs are below the actual costs for all four periods. This is hardly surprising, but it should be noted that the optimal decisions were derived with the levels of production and input utilization assumed. It should always be possible to find what more profitable and less expensive results might have been achieved, given the information that was gained by the occurrence of the event itself.

The reader may well regret that the activity levels are not given here for the optimal and actual costs, revenues and profits. This disappointment, however, cannot be rectified because of the conditions under which the data were obtained. The writer is pledged not to publish the actual data or optimal allocations. For the results obtained in this chapter, publication of the optimal allocations would be equivalent to identifying some of the actual data.

VII. CONCLUSIONS, EXTENSIONS, AND LIMITATIONS

The main emphasis in this study has been to construct and estimate a dynamic model for a multi-product, multi-purpose, multi-input firm. Inventory control functions were devised for both finished goods and materials and were connected by the production process. Two single equation regression techniques were applied in the estimation process; they were simple least squares and restricted least squares.

To indicate the role that the estimated model may play in decision making within the firm, two simplified optimization problems were solved in chapter six. These applications were consummated to show that the model could be utilized to order alternative production and inventory policies. Most economic cost studies conclude that inventory costs are relatively small when compared to the costs of producing the final product. This has been shown to be the case for this application, too. This, however, should not be taken to mean that the inventory considerations within the model are unnecessary. By the adaptive relationships connecting the changes in production and the level of purchases to output and input inventories, it is clear that the behavior and decision making within the firm is not independent of the inventory levels.

Charles Holt and his associates (46) studied a cost minimization problem in which production and the size of the work force were controlled variables and the inventory level was uncontrolled. In this thesis the levels of production and the use of labor, machine hours, and materials are fixed; the inventory levels for inputs and finished goods are computed. The learning functions for the changes in production of the ten products and the level of material purchases are similar to various applications suggested by A. W. Phillips (70) and Edwin Mills (59).

A great many assumptions have been made throughout this thesis. For the most part, each represents a limitation of this study. In chapter five the case was presented for not using simultaneous estimation techniques, but the reader who is not convinced will believe that the use of single equation methods is a major shortcoming of this effort. In chapter six the assumption that the output levels are predetermined reduces the significance of the optimal allocations greatly, but this presumption makes the results comparable to the actual costs and profits. Surely one would prefer to see the optimal costs and revenues computed for many more months; however, the determination of only four cases required a large number of computer hours.

Several other confines of this study may be viewed as extensions to be considered in future studies. Among these are: the construction of a long run model to replace the short run model studied here; the determination of optimal decisions, given a different objective such as sales maximization; the addition of constraints to the model to account for capital growth; the role of borrowing and holding funds, as was mentioned in chapter three; and the expansion of the model to include the accounting relationships that must balance within a firm.

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APPENDIX A

$$x^L = \left[\begin{array}{c} x_1^{L1} \\ \vdots \\ x_k^{L1} \\ \hline x_1^{L2} \\ \vdots \\ x_k^{L2} \\ \hline \vdots \\ \hline x_1^{Ln} \\ \vdots \\ x_k^{Ln} \end{array}\right] \qquad x^M = \left[\begin{array}{c} x_1^{M1} \\ \vdots \\ x_h^{M1} \\ \hline x_1^{M2} \\ \vdots \\ x_h^{M2} \\ \hline \vdots \\ \hline x_1^{Mn} \\ \vdots \\ x_h^{Mn} \end{array}\right] \qquad \bar{x} = \left[\begin{array}{c} \bar{x}_1^1 \\ \vdots \\ \bar{x}_m^1 \\ \hline \bar{x}_1^2 \\ \vdots \\ \bar{x}_m^2 \\ \hline \vdots \\ \hline \bar{x}_1^n \\ \vdots \\ \bar{x}_m^n \end{array}\right]$$

$$v = \left[\begin{array}{c} v_1^1 \\ \vdots \\ v_k^1 \\ \hline v_1^2 \\ \vdots \\ v_k^2 \\ \hline \vdots \\ \hline v_1^n \\ \vdots \\ v_k^n \end{array}\right] \qquad y = \left[\begin{array}{c} y_1^1 \\ \vdots \\ y_h^1 \\ \hline y_1^2 \\ \vdots \\ y_h^2 \\ \hline \vdots \\ \hline y_1^n \\ \vdots \\ y_h^n \end{array}\right] \qquad r = \left[\begin{array}{c} r_1^1 \\ \vdots \\ r_m^1 \\ \hline r_1^2 \\ \vdots \\ r_m^2 \\ \hline \vdots \\ \hline r_1^n \\ \vdots \\ r_m^n \end{array}\right]$$

$$A^L = \begin{bmatrix} a_1^{L1} \ldots a_k^{L1} & 0 & 0 & & \ldots & & 0 \ldots 0 \\ 0 \ \ldots 0 & a_1^{L2} \ldots a_k^{L2} & 0 & 0 \ldots & & & \vdots \quad \vdots \\ 0 \quad \ldots & & 0 & a_1^{L3} \ldots a_k^{L3} & 0 & \ldots & 0 \quad 0 \\ 0 \quad \ldots & & & 0 & \cdot & & \vdots \quad \vdots \\ \vdots & & & & & \cdot & 0 \ldots 0 \\ 0 \quad \ldots & & & & & 0 & a_1^{Ln} \ldots a_k^{Ln} \end{bmatrix}$$

$$A^M = \begin{bmatrix} a_1^{M1} \ldots a_h^{M1} & 0 \ldots & 0 & & & & 0 \ \ldots 0 \\ 0 \ldots 0 & a_1^{M2} \ldots a_h^{M2} & 0 \ldots & 0 & & & \vdots \quad \vdots \\ 0 \ . \ . \ . & & 0 & a_1^{M3} \ldots a_h^{M3} & 0 & & 0 \ \ldots 0 \\ \cdot & & & & & & \\ \cdot & & \cdot & 0 & \cdot & & \\ \cdot & & & & & \cdot & 0 \ \ldots 0 \\ 0 \ . \ . \ . & & & & & 0 & a_1^{Mn} \ldots a_1^{Mn} \end{bmatrix}$$

$$\overline{A} = \begin{bmatrix} a_1^1 \ldots a_m^1 \ 0 & 0 \ldots 0 & & \ldots & & 0 \ \ldots 0 \\ 0 \ldots 0 \ 0 & a_1^2 \ldots a_m^2 & 0 \ldots 0 & & & \vdots \quad \vdots \\ 0 \ . \ . \ . & 0 & a_1^3 \ldots a_m^3 \ 0 & & & 0 \ \ldots 0 \\ \cdot & & & & & \\ \cdot & \cdot & 0 & \cdot & & 0 \ \ldots 0 \\ \cdot & & & & \cdot & \\ 0 \ . \ . \ . & & & & 0 & a_1^n \ldots a_m^n \end{bmatrix}$$